Sensitivity equation method for the Navier—Stokes equations applied to uncertainty propagation

Séminaire du STMF

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Aim and context

AIM: develop an **uncertainty propagation tool** and integrate it into the TRUST code.

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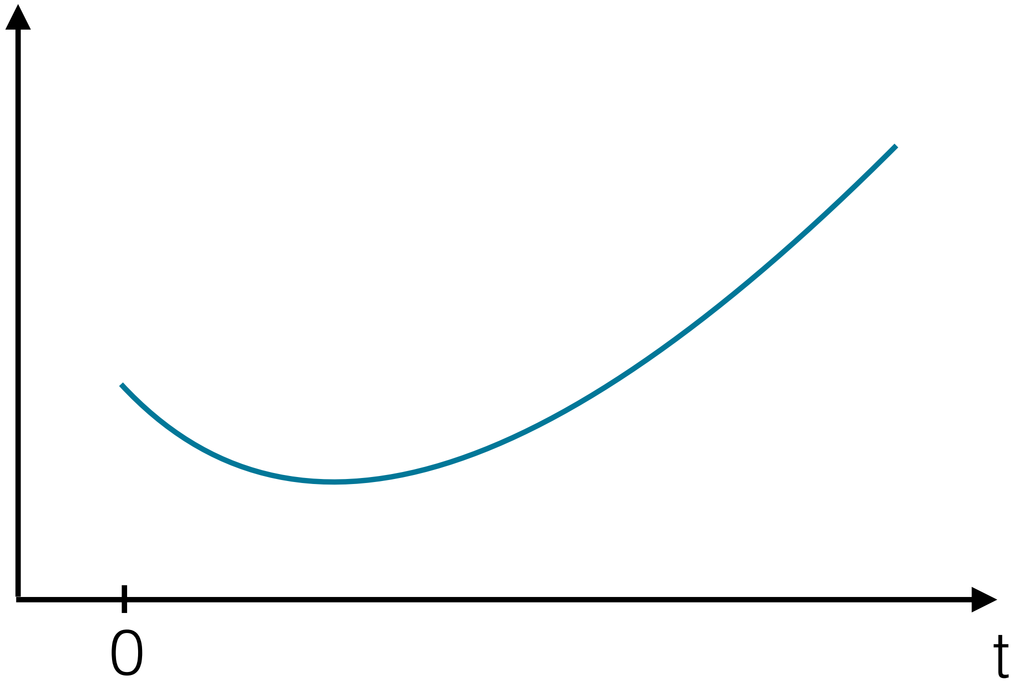

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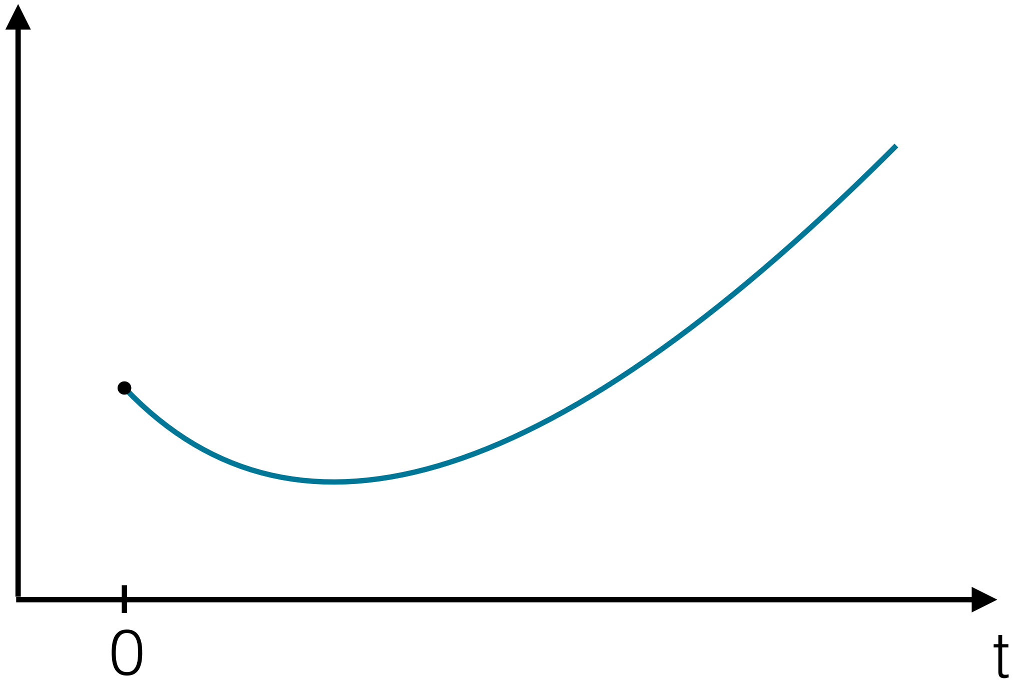

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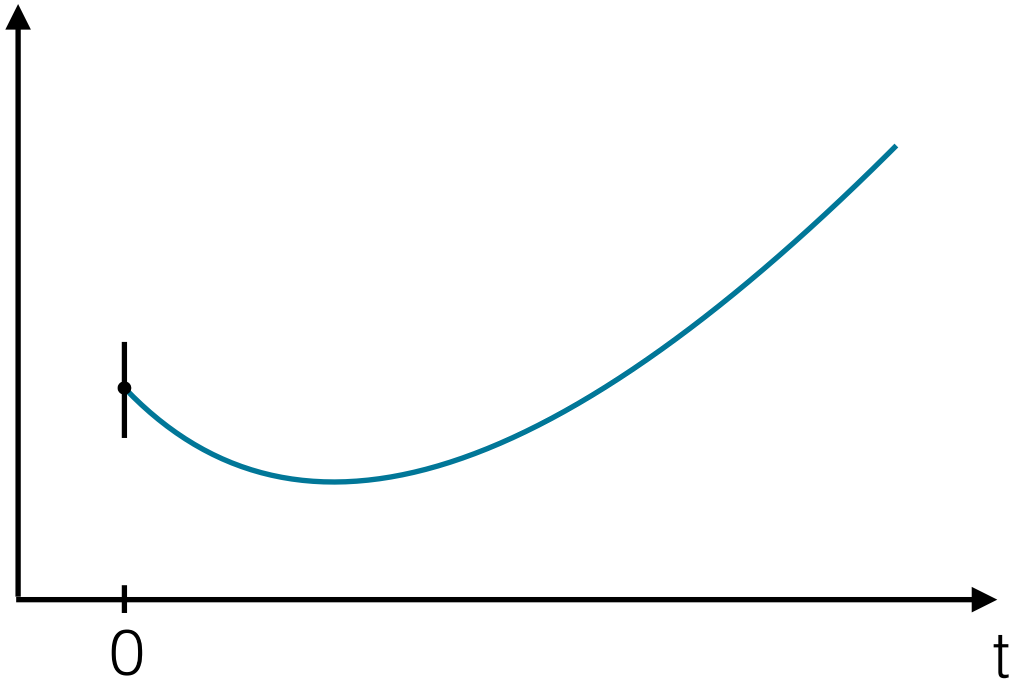
uncertain
initial condition



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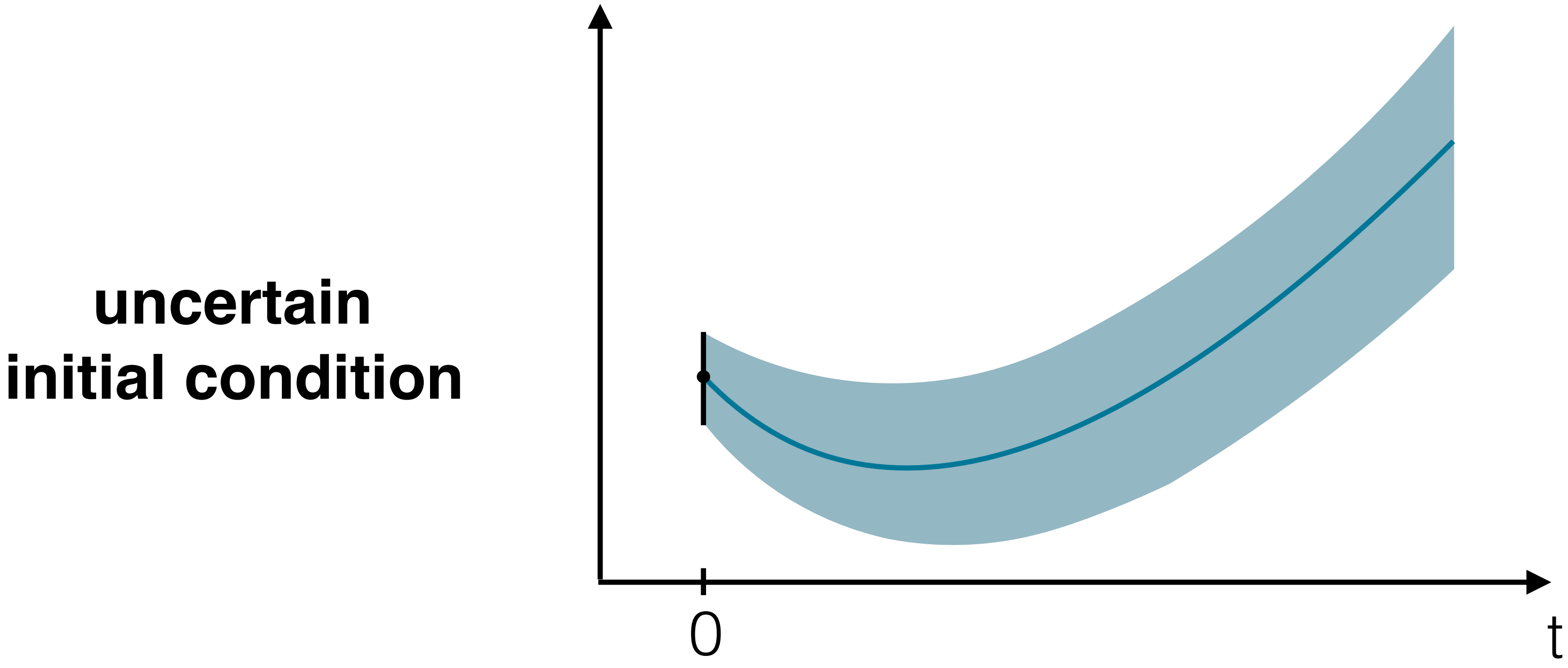

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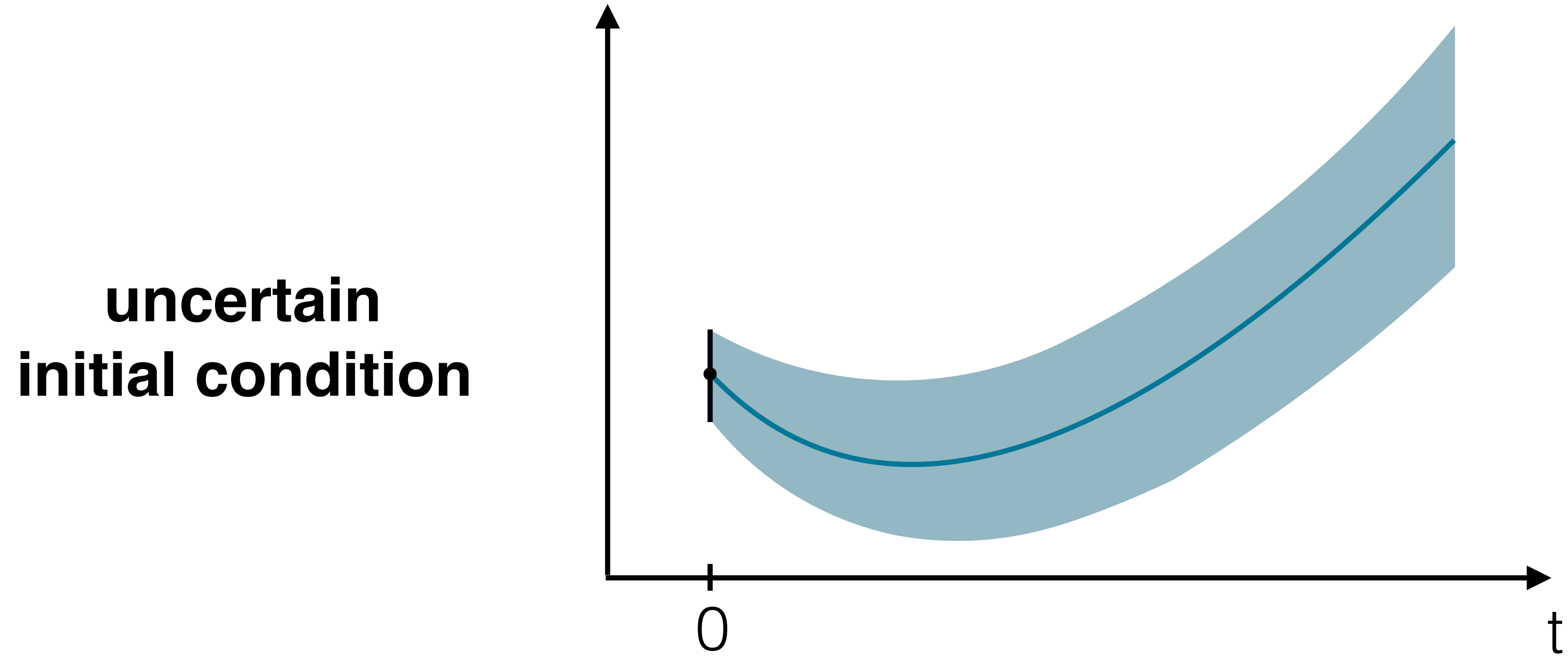

Ideal properties:

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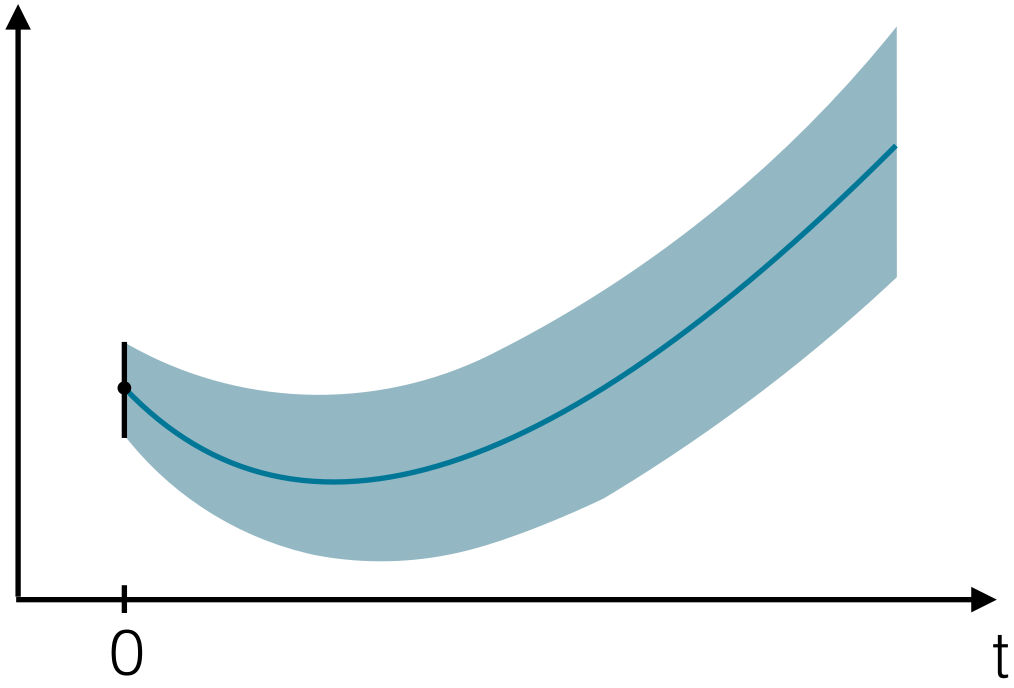

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Ideal properties:

- computationally affordable
- requires minimal code development

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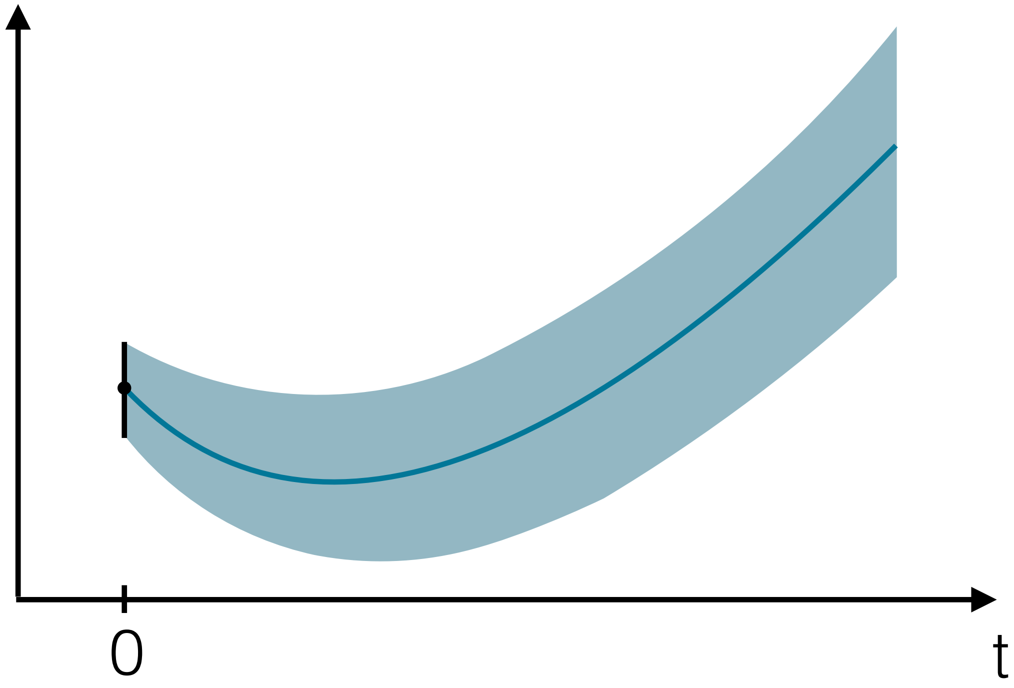

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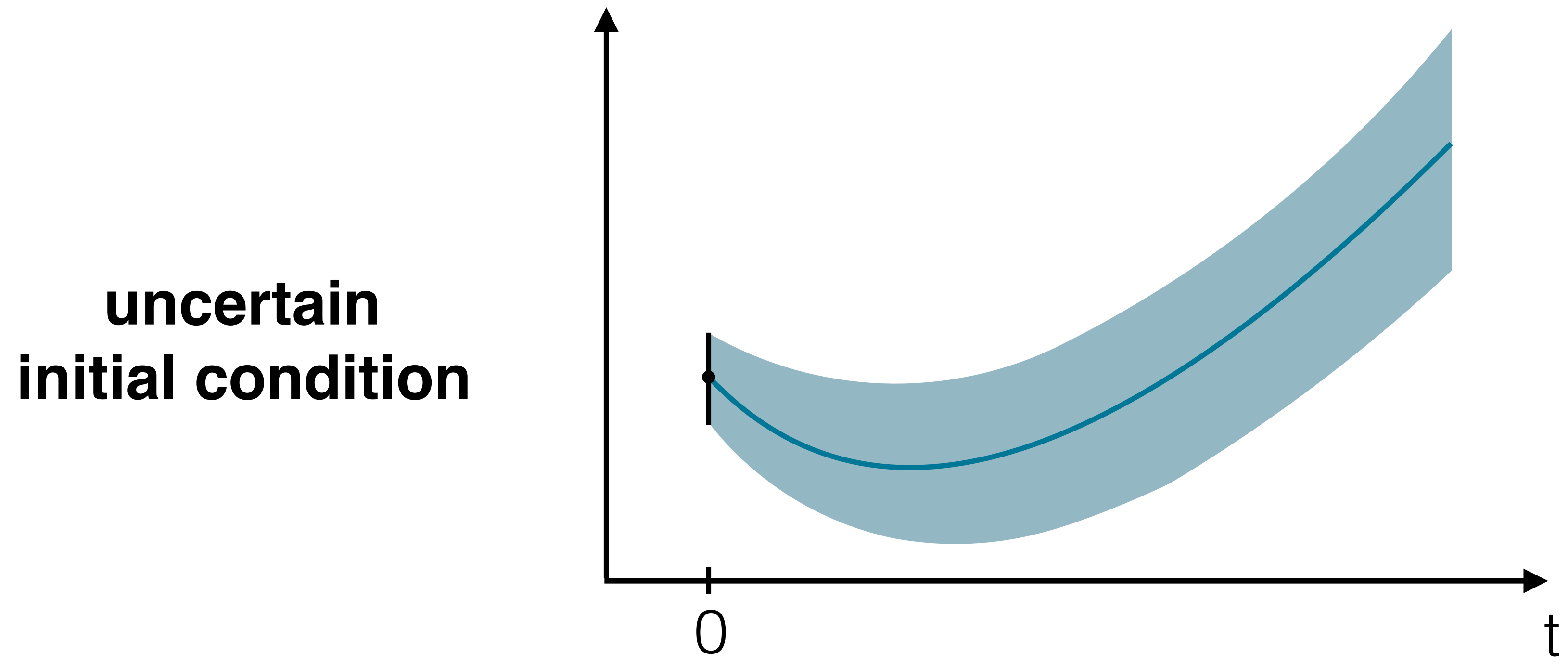

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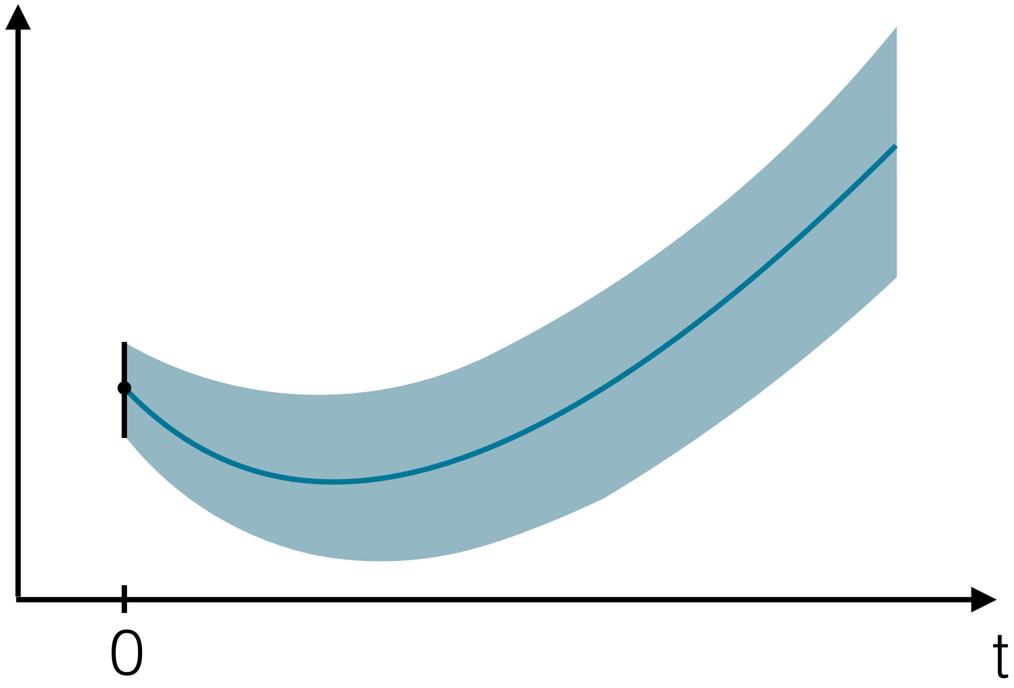

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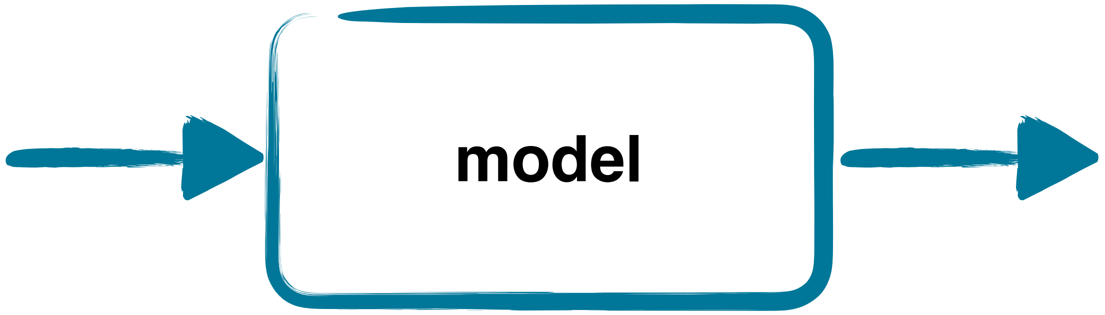
Sensitivity Analysis

Sensitivity analysis (SA) : study of how changes in the **inputs** of a model affect the **outputs**



input parameters

a

model

output (state)

$\mathbf{u}$

Sensitivity :

$$\frac{\partial \mathbf{u}}{\partial a} = \mathbf{u}_a$$

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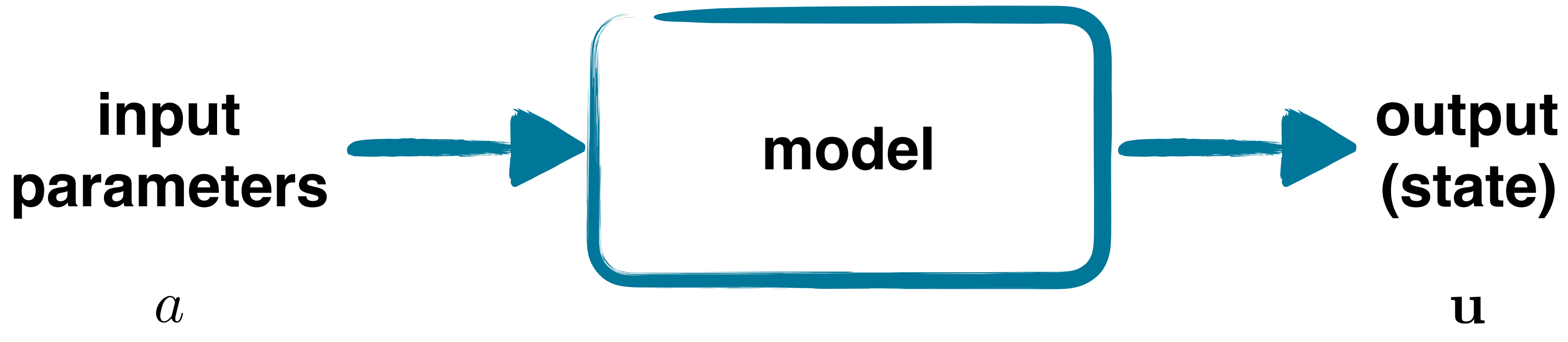

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$$\partial_t \mathbf{u} + \mathcal{L}(\mathbf{u}) = \mathbf{f} \quad \Omega, \ t > 0$$

+ initial and boundary conditions.

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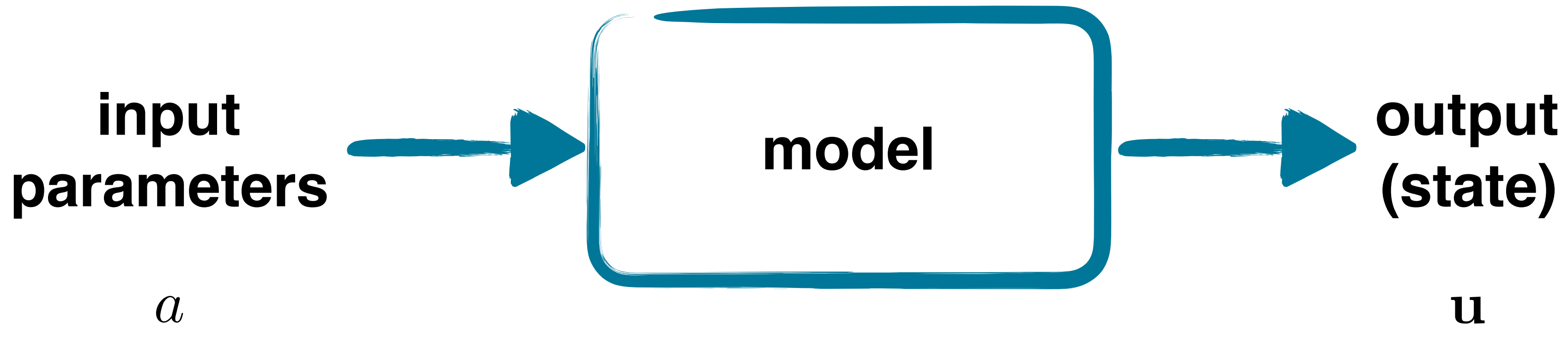

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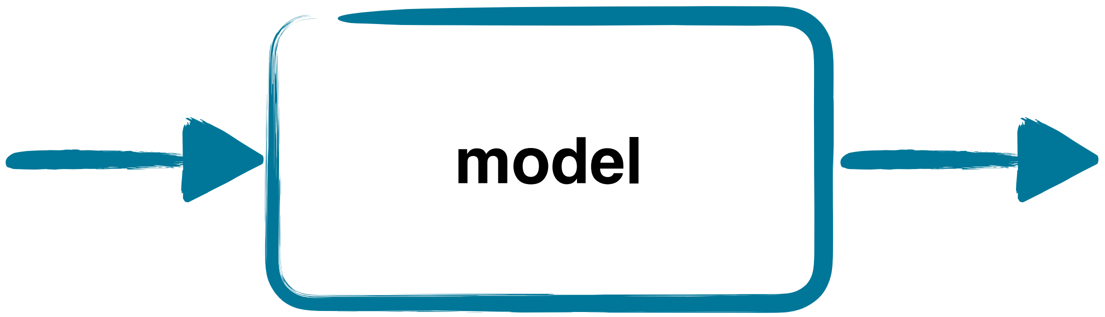

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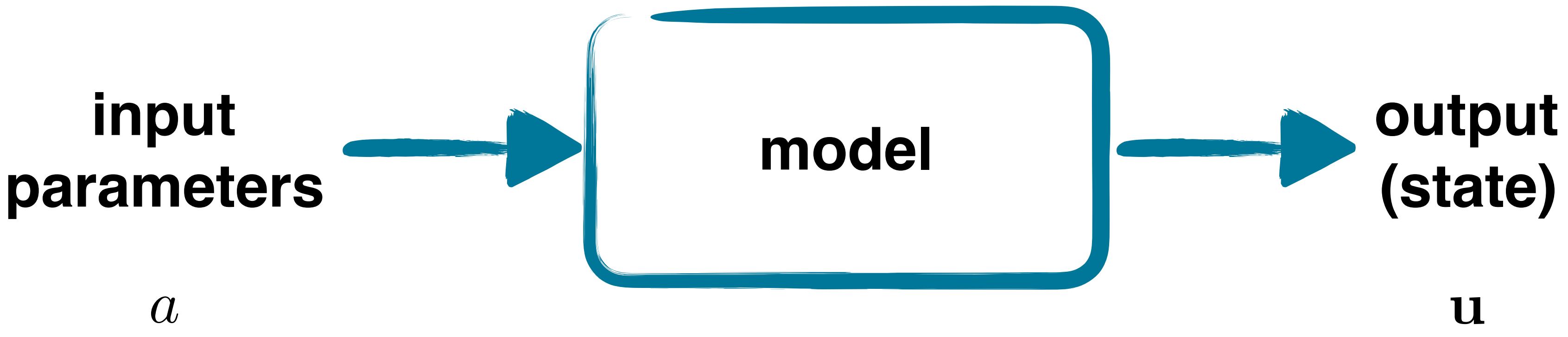

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State and sensitivity equations

The Navier—Stokes equations :

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} & \Omega, t > 0, \\ \nabla \cdot \mathbf{u} = 0 & \Omega, t > 0, \\ \mathbf{u}(\mathbf{x}, 0) = 0 & \Omega, t = 0, \\ \mathbf{u} = -g(y)\mathbf{n} & \text{on } \Gamma_{in}, \\ \mathbf{u} = 0 & \text{on } \Gamma_{w}, \\ (\nu \nabla \mathbf{u} - pI)\mathbf{n} = 0 & \text{on } \Gamma_{out}. \end{cases}$$

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Remark : these are known as the Oseen equations.

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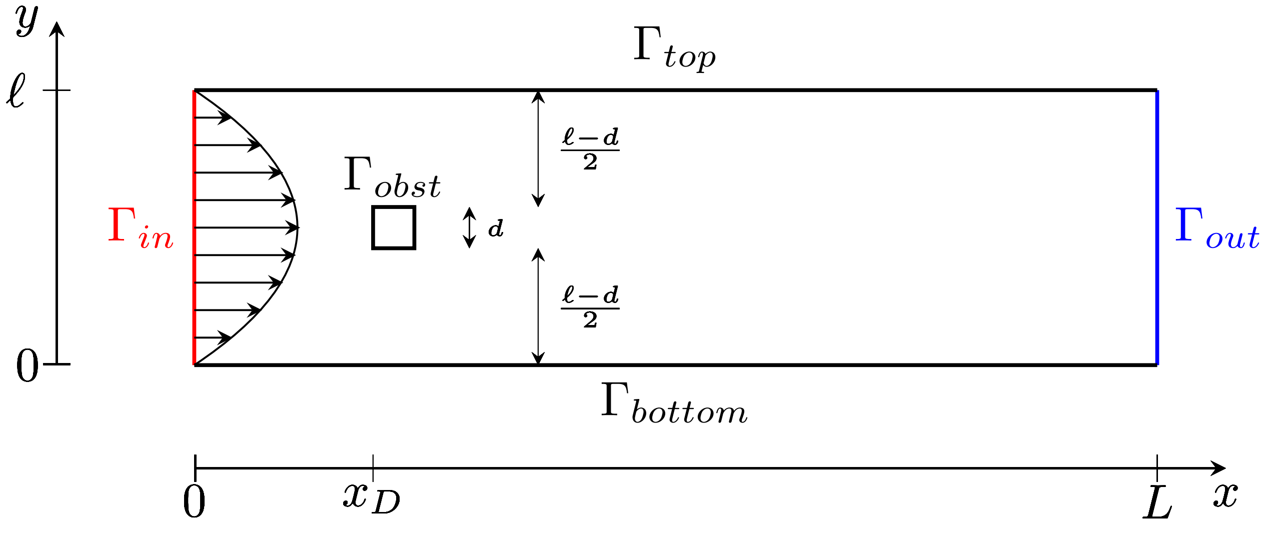
Domain :



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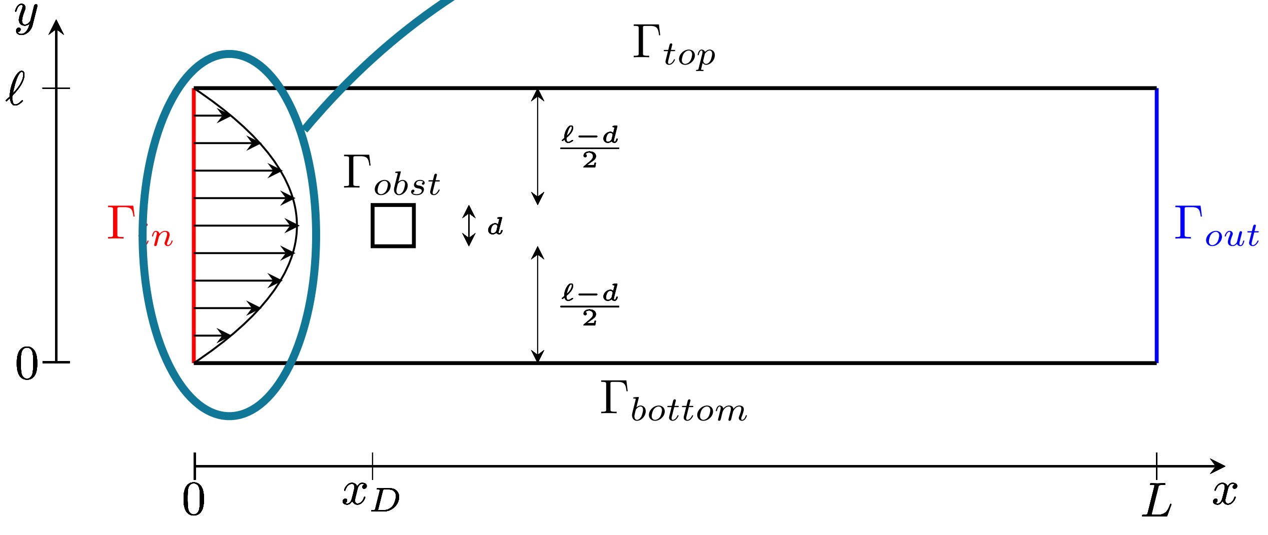
Domain :

$$g(y) = \frac{4A}{\ell^2} y(\ell - y)$$



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State stability

Proposition 1 *Let* $\mathbf{R}_g$ *be a sufficiently smooth stationary*[1] *function such that* $\nabla \cdot \mathbf{R}_g = 0$ *in* Ω*,* $\mathbf{R}_g = \mathbf{u}$ *on* $\Gamma_{in} \cup \Gamma_w$*, and* $\nabla \mathbf{R}_g \mathbf{n}|_{\Gamma_{out}} = 0$*. Then, if* $\mathbf{u} \cdot \mathbf{n}| \geqslant 0$ *on* Γ_{out} *and* $\tilde{\mathbf{f}}$ *is stationary the following stability estimate holds:*

$$|||\mathbf{u}|||^2 \leqslant \|\mathbf{R}_g\|^2 + \|\tilde{\mathbf{f}}(t)\|^2 t + K(\mathbf{R}_g, \tilde{\mathbf{f}}) e^{2t\|\nabla \mathbf{R}_g\|_{L^\infty}}, \qquad (1)$$

where $\tilde{\mathbf{f}} = \mathbf{f} + \nu \Delta \mathbf{R}_g - (\mathbf{R}_g \cdot \nabla)\mathbf{R}_g$*, and the norm* $|||\cdot|||$ *is defined as follows:*

$$|||\mathbf{u}|||^2 := \|\mathbf{u}(T)\|^2 + 2\nu \int_0^T \|\nabla \mathbf{u}(t)\|^2 dt.$$

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[1] If the boundary conditions depend on time, $\mathbf{R}_g$ cannot be taken stationary.

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$L^2 - \text{norm}$

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$$\exists \kappa = \kappa(\mathbf{u}, \Omega) > 0 \; : \; \left| \int_\Omega [(\mathbf{u}_a \cdot \nabla)\mathbf{u}] \cdot \mathbf{u}_a \right| \leqslant \kappa \|\mathbf{u}_a\|^2,$$

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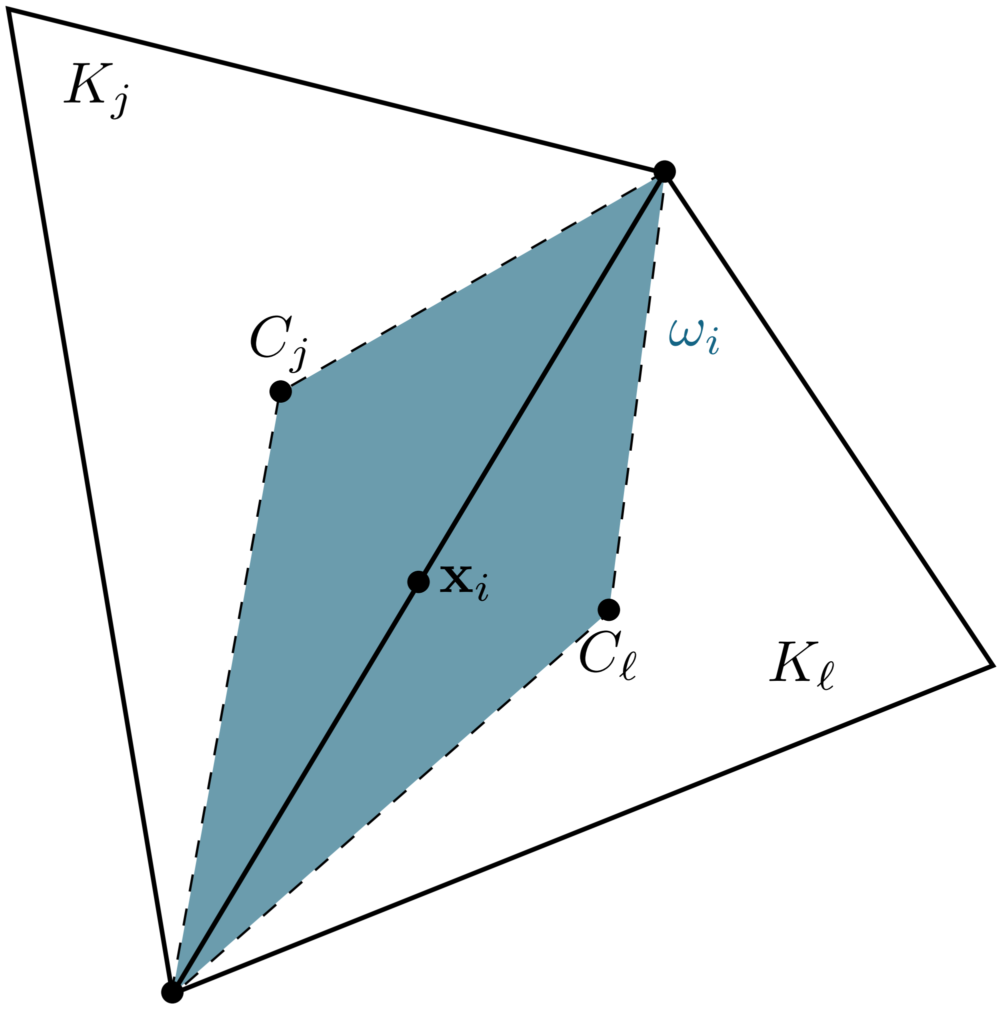
“Morally” it is $\|\nabla \mathbf{u}\|_\infty$

Spatial discretisation

The code TRUST TrioCFD is based on a **finite elements volumes** method (FEV).



Ingredients:

$\mathcal{T}_h$ triangulation of the domain Ω

$K_j \in \mathcal{T}_h$ triangles $j = 1, \ldots, N_T$

$\mathbf{x}_i$ nodes $i = 1, \ldots, N_N$

ω_i control volume

Spaces:

$$Q_h = \{q_h : \forall K \in \mathcal{T}_h,\ q_h|_K \in P_0(K)\},$$
$$V_h = \{w_h \text{ continuous in } \mathbf{x}_i : \forall K \in \mathcal{T}_h,\ w_h|_K \in P_1(K)\},$$
$$\mathbf{V}_h = \{\mathbf{w}_h = (w_x, w_y)^t : w_x,\ w_y \in V_h\}.$$

Basis functions : $\varphi_i(\mathbf{x}_j) = \delta_{i,j}$ for V_h
χ_K for Q_h

Remark : $V_h \not\subset H^1(\Omega)$

Spatial discretisation

We integrate the mass equation and its sensitivity over the triangles and the momentum equation and its sensitivity over the control volumes :

$$\begin{aligned}
\int_{\partial K_j \setminus \Gamma_D} \mathbf{u}_h \cdot \mathbf{n} = -\int_{\partial K_j \cap \Gamma_D} \mathbf{u}_h \cdot \mathbf{n} \quad & \forall K_j \in \mathcal{T}_h, \\
-\int_{\partial \omega_i \setminus \Gamma_N} (\nu \nabla \mathbf{u}_h - p_h I)\mathbf{n} + \int_{\partial \omega_i} (\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n} = \int_{\omega_i} \mathbf{f} \quad & \forall \omega_i, \\
\int_{\partial K_j \setminus \Gamma_D} \mathbf{u}_{a,h} \cdot \mathbf{n} = -\int_{\partial K_j \cap \Gamma_D} \mathbf{u}_{a,h} \cdot \mathbf{n} \quad & \forall K_j \in \mathcal{T}_h, \\
-\int_{\partial \omega_i \setminus \Gamma_N} (\nu \nabla \mathbf{u}_{a,h} - p_{a,h} I)\mathbf{n} + \int_{\partial \omega_i} (\mathbf{u}_{a,h} \otimes \mathbf{u}_h + \mathbf{u}_h \otimes \mathbf{u}_{a,h})\mathbf{n} = \int_{\omega_i} \overline{\mathbf{f}}_a \quad & \forall \omega_i.
\end{aligned}$$

Using the basis functions of the spaces previously introduced one has :

$$\begin{aligned}
\mathbf{u}_h(\mathbf{x}) &= \sum_{i=1}^{N_N} \mathbf{u}_h(\mathbf{x}_i)\varphi_i(\mathbf{x}), & p_h(\mathbf{x}) &= \sum_{j=1}^{N_T} p_h(K_j)\chi_{K_j}, \\
\mathbf{u}_{a,h}(\mathbf{x}) &= \sum_{i=1}^{N_N} \mathbf{u}_{a,h}(\mathbf{x}_i)\varphi_i(\mathbf{x}), & p_{a,h}(\mathbf{x}) &= \sum_{j=1}^{N_T} p_{a,h}(K_j)\chi_{K_j}.
\end{aligned}$$

Spatial discretisation

$$\sum_{i\in\mathcal{I}\cup\mathcal{N}} \mathbf{u}_h(\mathbf{x}_i)\cdot\int_{\partial K_j\setminus\Gamma_D}\varphi_i\mathbf{n} = -\sum_{i\in\mathcal{D}}\mathbf{u}_h(\mathbf{x}_i)\cdot\int_{\partial K_j\cap\Gamma_D}\varphi_i\mathbf{n} \qquad \forall K_j\in\mathcal{T}_h,$$

$$-\sum_{j\in\mathcal{I}}\int_{\partial\omega_i}(\nu\mathbf{u}_h(\mathbf{x}_j)\otimes\nabla\varphi_j)\,\mathbf{n} + \sum_{j=1}^{N_T} p_h(K_j)\int_{\partial\omega_i}\chi_{K_j}\mathbf{n} + \int_{\partial\omega_i}\mathbf{u}_h(\mathbf{u}_h\cdot\mathbf{n}) = \int_{\omega_i}\mathbf{f} \qquad \forall\omega_i,$$

$$\sum_{i\in\mathcal{I}\cup\mathcal{N}} \mathbf{u}_{a,h}(\mathbf{x}_i)\cdot\int_{\partial K_j\setminus\Gamma_D}\varphi_i\mathbf{n} = -\sum_{i\in\mathcal{D}}\mathbf{u}_{a,h}(\mathbf{x}_i)\cdot\int_{\partial K_j\cap\Gamma_D}\varphi_i\mathbf{n} \qquad \forall K_j\in\mathcal{T}_h,$$

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Spatial discretisation

$$\sum_{i\in\mathcal{I}\cup\mathcal{N}} \overset{\mathbf{U}_i}{\boxed{\mathbf{u}_h(\mathbf{x}_i)}}\cdot\overset{B_{j,i}}{\boxed{\int_{\partial K_j\setminus\Gamma_D}\varphi_i\mathbf{n}}} = \overset{D_j}{\boxed{-\sum_{i\in\mathcal{D}}\mathbf{u}_h(\mathbf{x}_i)\cdot\int_{\partial K_j\cap\Gamma_D}\varphi_i\mathbf{n}}} \qquad \forall K_j\in\mathcal{T}_h,$$

$$-\sum_{j\in\mathcal{I}} \overset{A_{i,j}}{\boxed{\int_{\partial\omega_i}(\nu\mathbf{u}_h(\mathbf{x}_j)\otimes\nabla\varphi_j)\,\mathbf{n}}} + \sum_{j=1}^{N_T}\overset{P_j}{\boxed{p_h(K_j)}}\overset{C_{i,j}}{\boxed{\int_{\partial\omega_i}\chi_{K_j}\mathbf{n}}} + \overset{[L(\mathbf{U})\mathbf{U}]_i}{\boxed{\int_{\partial\omega_i}\mathbf{u}_h(\mathbf{u}_h\cdot\mathbf{n})}} = \overset{F_i}{\boxed{\int_{\omega_i}\mathbf{f}}} \qquad \forall\omega_i,$$

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$$-\sum_{j\in\mathcal{I}} \overset{A_{i,j}}{\boxed{\int_{\partial\omega_i}(\nu\mathbf{u}_{a,h}(\mathbf{x}_j)\otimes\nabla\varphi_j)\,\mathbf{n}}} + \sum_{j=1}^{N_T}\overset{P_{a,j}}{\boxed{p_{a,h}(K_j)}}\overset{C_{i,j}}{\boxed{\int_{\partial\omega_i}\chi_{K_j}\mathbf{n}}} + \overset{[L(\mathbf{U}_a)\mathbf{U}+L(\mathbf{U})\mathbf{U}_a]_i}{\boxed{\int_{\partial\omega_i}\mathbf{u}_{a,h}(\mathbf{u}_h\cdot\mathbf{n})+\mathbf{u}_h(\mathbf{u}_{a,h}\cdot\mathbf{n})}} = \overset{F_{a,i}}{\boxed{\int_{\omega_i}\bar{\mathbf{f}}_a}} \quad \forall\omega_i,$$

Spatial discretisation

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Spatial discretisation

$$
\begin{gathered}
A\mathbf{U} + B^t P + L(\mathbf{U})\mathbf{U} = \mathbf{F} \\
B\mathbf{U} = D \\
A\mathbf{U}_a + B^t P_a + L(\mathbf{U}_a)\mathbf{U} + L(\mathbf{U})\mathbf{U}_a = \mathbf{F}_a \\
B\mathbf{U}_a = D_a
\end{gathered}
$$

Spatial discretisation

$$
\begin{gathered}
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\end{gathered}
$$

Time discretisation

A number of different time schemes are implemented in the code TRUST TrioCFD:

- Forward Euler
- Runge Kutta
  - 2nd, 3rd and 4th order
- Semi-implicit Euler
  - implicit diffusion
  - explicit convection

Time discretisation

We use forward Euler :

$$\begin{cases} M\frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\Delta t} = -A\mathbf{U}^n - L(\mathbf{U}^n)\mathbf{U}^n - B^t P^{n+1} + F^n \\ B\mathbf{U}^{n+1} = 0 \end{cases}$$

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$$M\frac{\mathbf{U}^* - \mathbf{U}^n}{\Delta t} = -A\mathbf{U}^n - L(\mathbf{U}^n)\mathbf{U}^n - B^t P^n + F^n$$

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$$(BM^{-1}B^t)\delta P = \frac{1}{\Delta t}B\mathbf{U}^*$$

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$$\mathbf{U}^{n+1} = \mathbf{U}^{*} - \Delta t M^{-1}B^{t}\delta P$$

Time discretisation

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The same procedure is used for the sensitivity equations.

.data file

.data file

```
Pb_Hydraulique pb_etat
```

.data file

```
Pb_Hydraulique pb_etat
Pb_Hydraulique_sensibility pb_sensibilite
```

.data file

```
Pb_Hydraulique pb_etat
Pb_Hydraulique_sensibility pb_sensibilite
...
Probleme_Couple pbc
Associate pbc pb_etat
Associate pbc pb_sensibilite
```

.data file

```
Pb_Hydraulique pb_etat
Pb_Hydraulique_sensibility pb_sensibilite
...
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```

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```
Pb_Hydraulique pb_etat
Pb_Hydraulique_sensibility pb_sensibilite
...
Probleme_Couple pbc
Associate pbc pb_etat
Associate pbc pb_sensibilite
...
Read pb_sensibilite
{
  Navier_Stokes_standard_sensibility
  {
```

.data file

```
Pb_Hydraulique pb_etat
Pb_Hydraulique_sensibility pb_sensibilite
...
Probleme_Couple pbc
Associate pbc pb_etat
Associate pbc pb_sensibilite
...
Read pb_sensibilite
{
 Navier_Stokes_standard_sensibility
 {
  uncertain_variable { vitesse }
```

.data file

```
Pb_Hydraulique pb_etat
Pb_Hydraulique_sensibility pb_sensibilite
...
Probleme_Couple pbc
Associate pbc pb_etat
Associate pbc pb_sensibilite
...
Read pb_sensibilite
{
 Navier_Stokes_standard_sensibility
 {
   uncertain_variable { vitesse }
   state { pb_champ_evaluateur pb_etat vitesse }
```

.data file

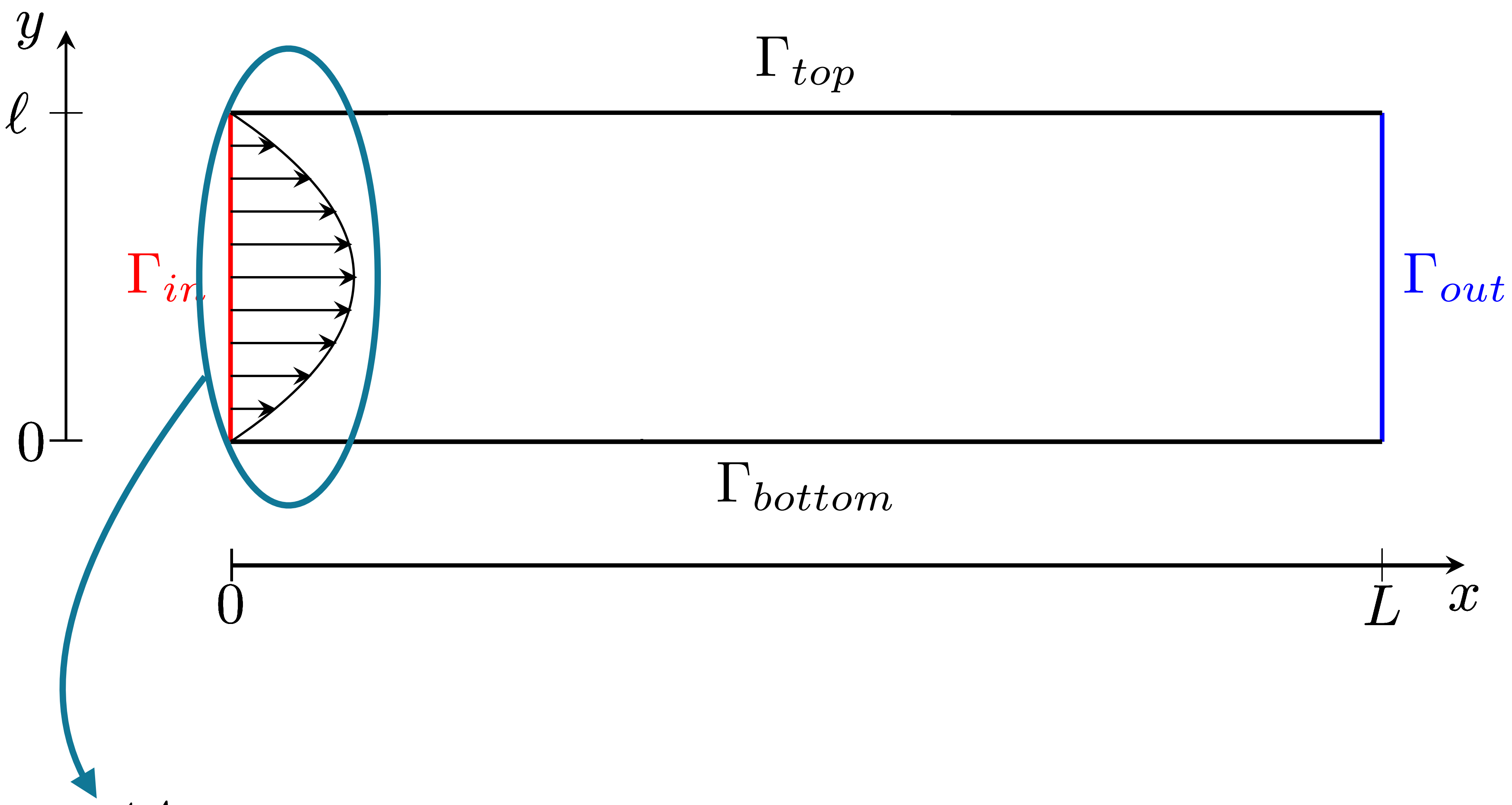
```
Pb_Hydraulique pb_etat
Pb_Hydraulique_sensibility pb_sensibilite
...
Probleme_Couple pbc
Associate pbc pb_etat
Associate pbc pb_sensibilite
...
Read pb_sensibilite
{
 Navier_Stokes_standard_sensibility
 {
   uncertain_variable { vitesse }
   state { pb_champ_evaluateur pb_etat vitesse }
   convection { Sensibility { amont } }
```

.data file

```
Pb_Hydraulique pb_etat
Pb_Hydraulique_sensibility pb_sensibilite
...
Probleme_Couple pbc
Associate pbc pb_etat
Associate pbc pb_sensibilite
...
Read pb_sensibilite
{
  Navier_Stokes_standard_sensibility
  {
    uncertain_variable { vitesse }
    state { pb_champ_evaluateur pb_etat vitesse }
    convection { Sensibility { amont } }
    …
    boundary_conditions
    {
      inlet frontiere_ouverte_vitesse_imposee Champ_Front_fonc_xyz 2 Y*(0.7-Y)/0.35 0.
      ...
    }
    ...
  }
}
```

Test case description

Domain :

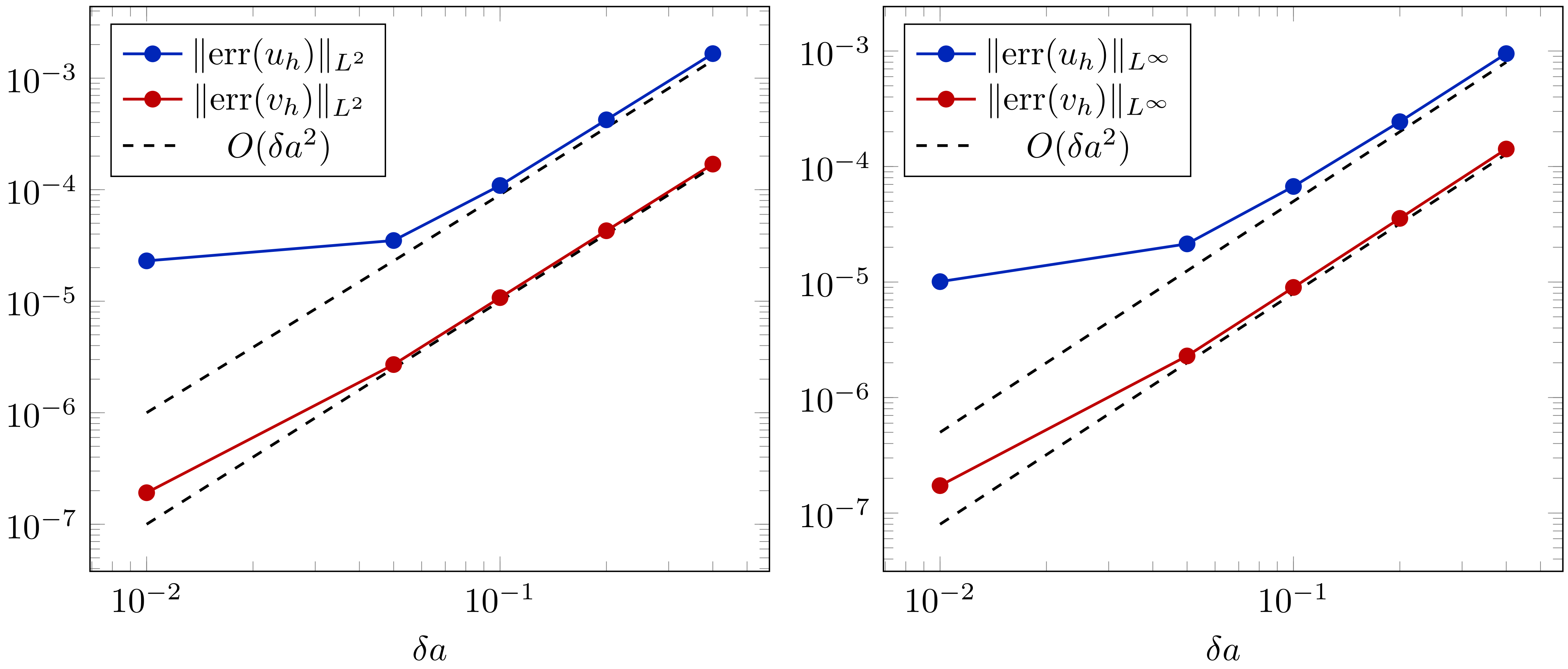


$g(y) = \frac{4A}{\ell^2} y(\ell - y)$ Uncertain parameter $a = A$

Validation of the CSE method

Results

$$\mathrm{err}(\mathbf{u}) = \mathbf{u}(x,T;a+\delta a) - \mathbf{u}(x,T;a) - \delta a \mathbf{u}_a(x,T;a) \simeq O(\delta a^2)$$



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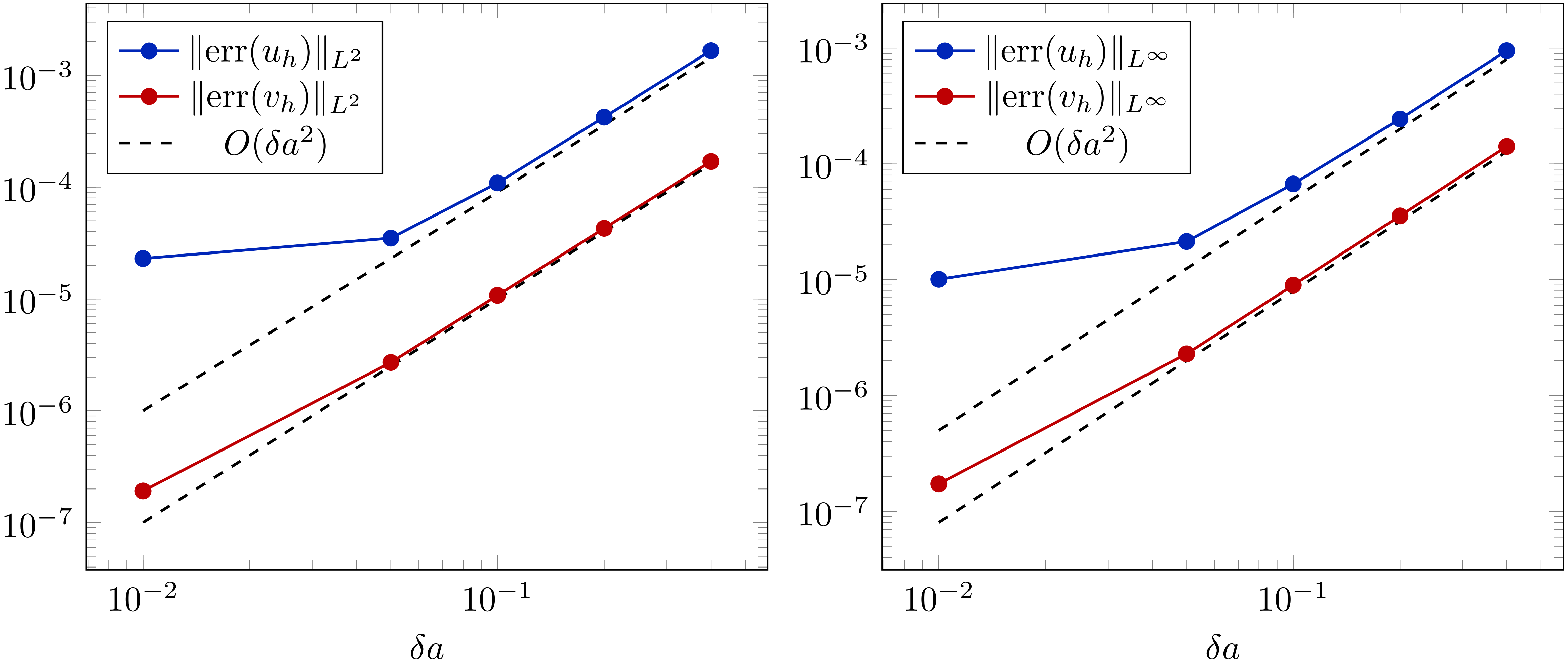

$$O(\delta a^2) \simeq O(h^r)$$

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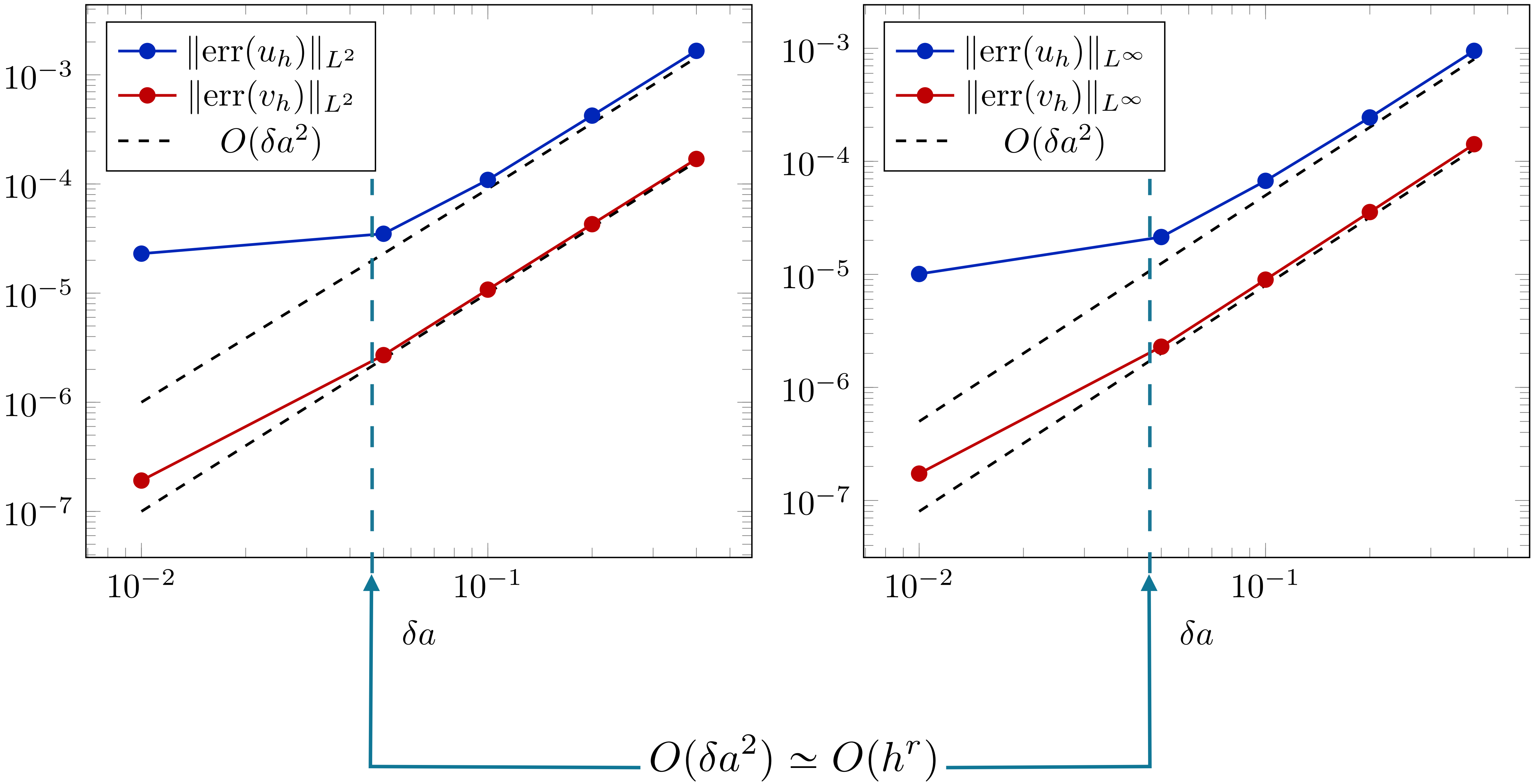

$O(\delta a^2) \simeq O(h^r)$

Uncertainty quantification

Let **a** be a gaussian random vector, with the following average and variance:

$$\mu_{\mathbf{a}} = \begin{bmatrix} \mu_{a_1} \\ \vdots \\ \mu_{a_M} \end{bmatrix}, \qquad \sigma_{\mathbf{a}} = \begin{bmatrix} \sigma^2_{a_1} & \mathrm{cov}(a_1, a_2) & \dots & \mathrm{cov}(a_1, a_M) \\ \mathrm{cov}(a_1, a_2) & \sigma^2_{a_2} & \dots & \mathrm{cov}(a_2, a_M) \\ \vdots & & \ddots & \vdots \\ \mathrm{cov}(a_1, a_M) & \dots & & \sigma^2_{a_M} \end{bmatrix},$$

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Aim: determine a **confidence interval**

$$CI_X = [\mu_X - d(\sigma_X), \mu_X + d(\sigma_X)]$$
$$P(X \in CI_X) \geqslant 1 - \alpha$$

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Monte Carlo approach: *N* samples of the state X_k

$$\mu_X = \frac{1}{N}\sum_{k=1}^{N} X_k \qquad \sigma^2_X = \frac{1}{N-1}\sum_{k=1}^{N} |\mu_X - X_k|^2$$

Uncertainty quantification

Sensitivity approach: let us consider the following first order Taylor expansion

$$X(\mathbf{a}) = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M} (a_i - \mu_{a_i}) X_{a_i}(\mu_{\mathbf{a}}) + o(\|\mathbf{a} - \mu_{\mathbf{a}}\|^2)$$

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Then, computing, the average one has:

$$\mu_X = E[X(\mathbf{a})] = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}})E[a_i - \mu_{a_i}] = X(\mu_{\mathbf{a}}),$$

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And for the variance:

$$\sigma_X^2 = E[(X(\mathbf{a}) - \mu_X)^2] = E\left[\left(\sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}})(a_i - \mu_{a_i})\right)^2\right] =$$

$$= \sum_{i=1}^{M} X_{a_i}^2(\mu_{\mathbf{a}}) E[(a_i - \mu_{a_i})^2] + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i}(\mu_{\mathbf{a}}) X_{a_j}(\mu_{\mathbf{a}}) E[(a_i - \mu_{a_i})(a_j - \mu_{a_j})].$$

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Therefore, we have the following first order estimates:

$$\mu_X = X(\mu_{\mathbf{a}}), \quad \sigma_X^2 = \sum_{i=1}^{M} X_{a_i}^2 \sigma_{a_i}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i} X_{a_j} \mathrm{cov}(a_i, a_j).$$

Uncertainty quantification

Monte Carlo approach:

👍

Provides information on the distribution of the output

Smaller confidence intervals

👎

Computational cost

SA approach :

👍

Reasonable computational cost (depending on the number of parameters)

👎

No insight on the distribution of the output

Valid for small variation of the input

Uncertainty quantification

How to compute a confidence interval for an unknown distribution?

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Chebyshev's inequality: $$P(|X - \mu_X| \geqslant \lambda) \leqslant \frac{\sigma_X^2}{\lambda^2} \quad \forall \lambda > 0$$

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Which implies: $P(X \in (\mu_X - \lambda, \mu_X + \lambda)) \geqslant 1 - \frac{\sigma_X^2}{\lambda^2}.$

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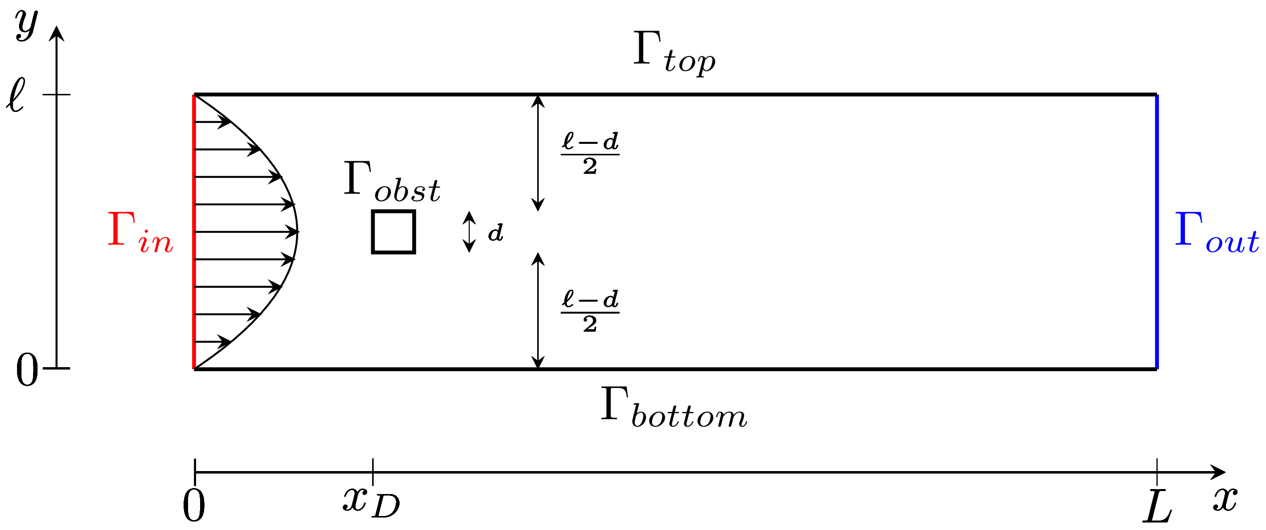
Which implies: $P(X \in (\mu_X - \lambda, \mu_X + \lambda)) \geqslant 1 - \dfrac{\sigma_X^2}{\lambda^2}.$

Now, by asking for $1 - \dfrac{\sigma_X^2}{\lambda^2} = 1 - \alpha$ one obtains $\lambda = \dfrac{\sigma_X}{\sqrt{\alpha}}.$

Remark: if the variable X was gaussian, to obtain a 95% confidence interval the half amplitude should be $1.96\sigma_X$, whilst with this method we obtain $4.47\sigma_X$.

Steady test case

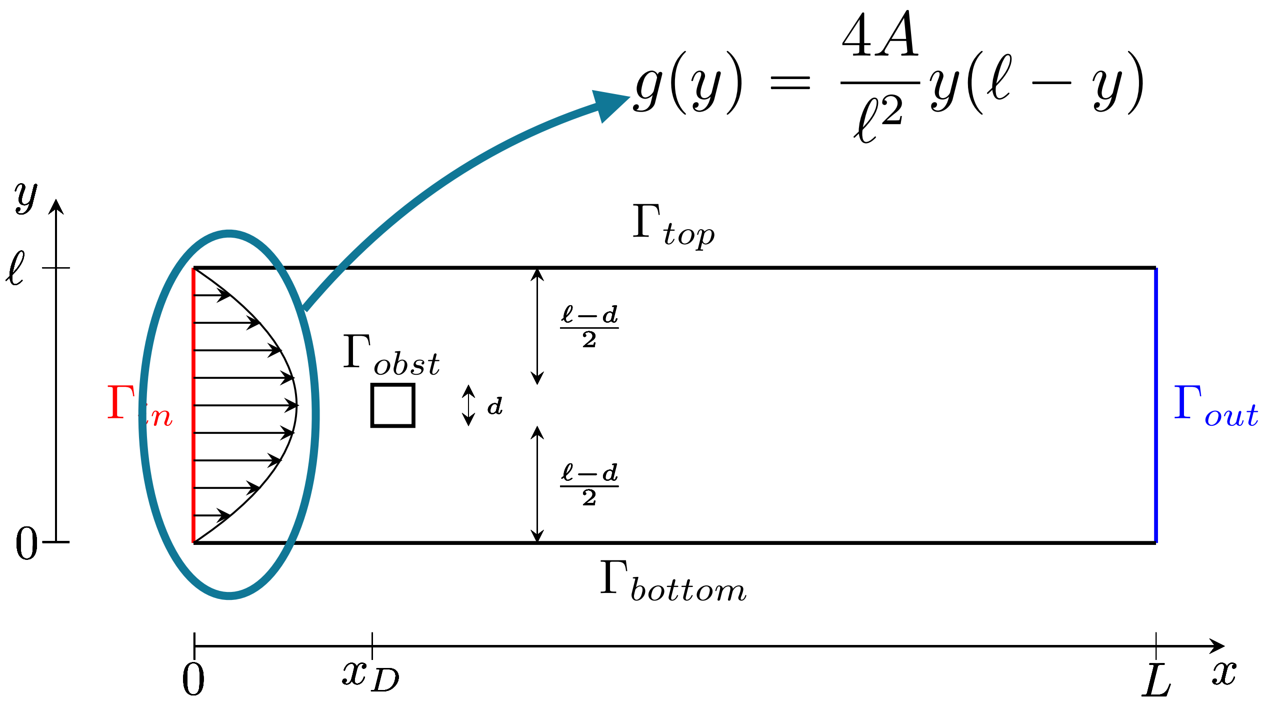
Domain :



$$\Gamma_w = \Gamma_{obst} \cup \Gamma_{top} \cup \Gamma_{bottom}$$

Steady test case

Domain :



$$g(y) = \frac{4A}{\ell^2} y(\ell - y)$$

$$\Gamma_w = \Gamma_{obst} \cup \Gamma_{top} \cup \Gamma_{bottom}$$

Steady test case

Domain :

uncertain parameter

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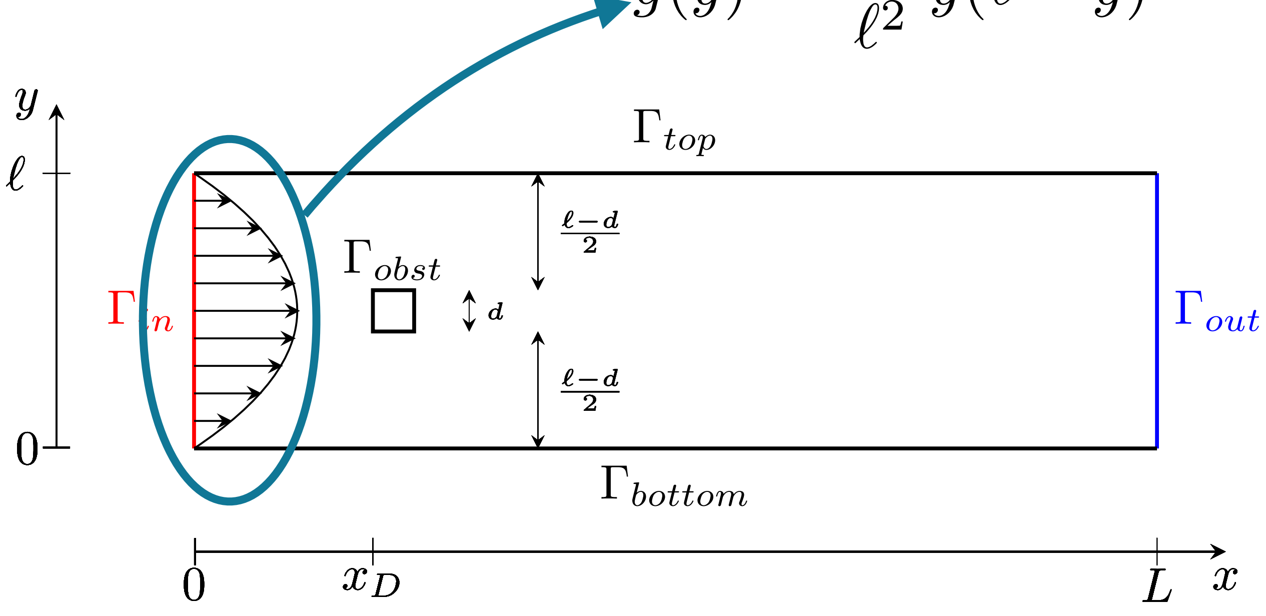

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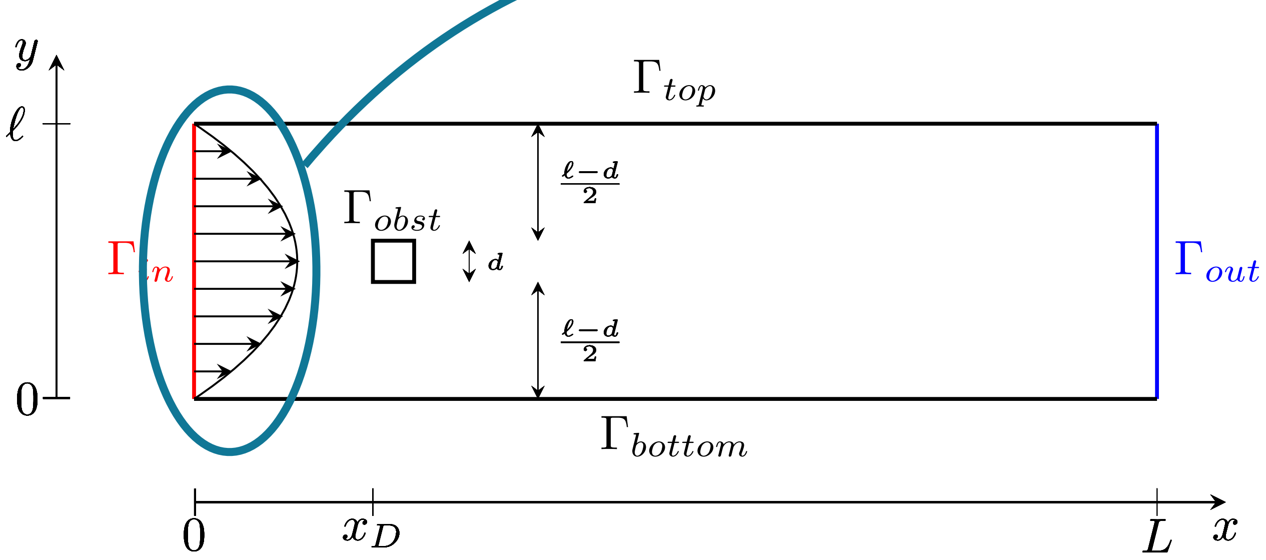

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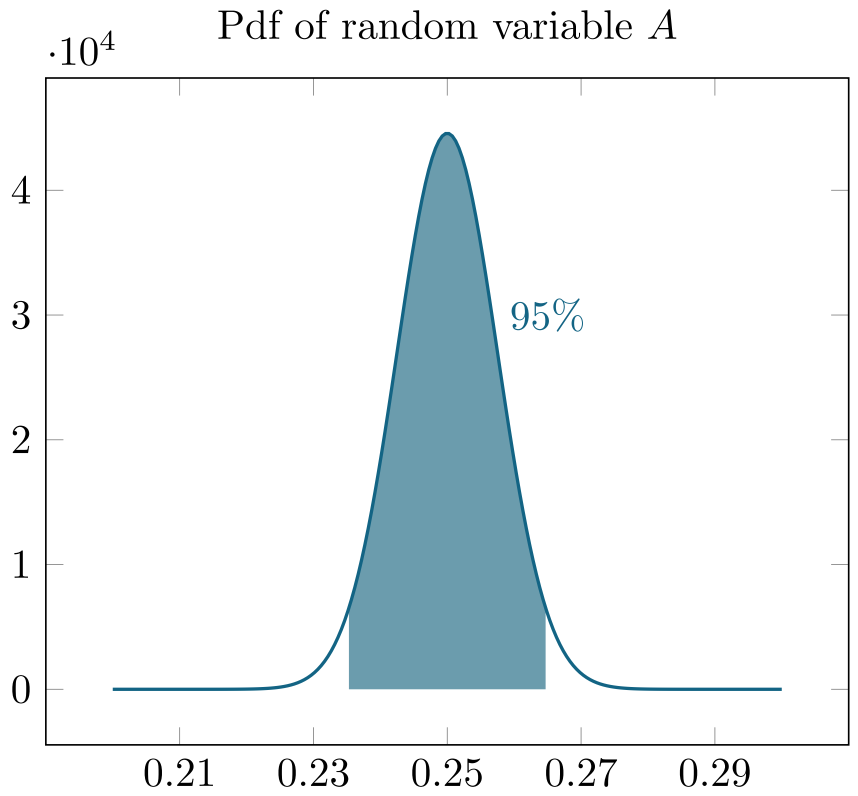
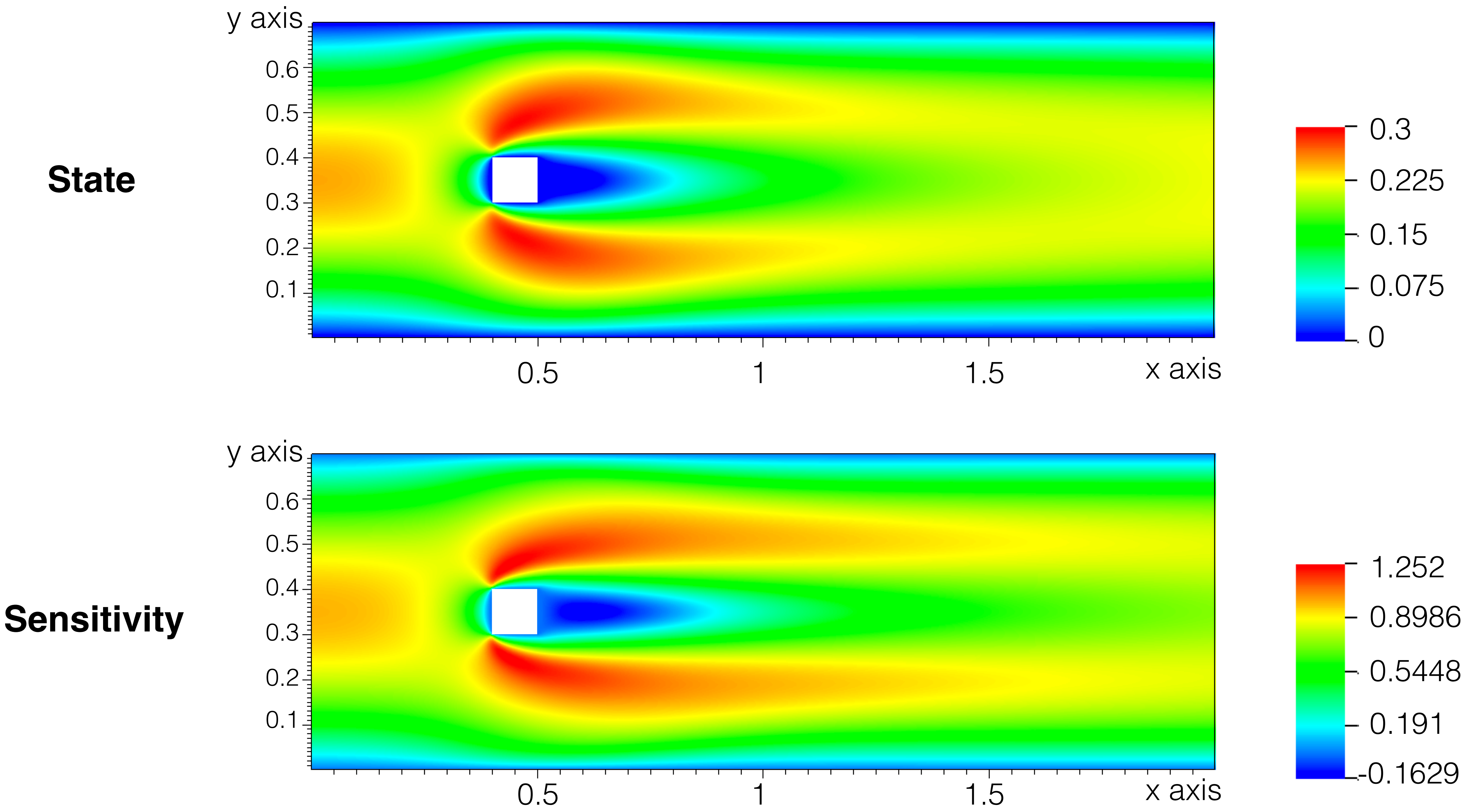


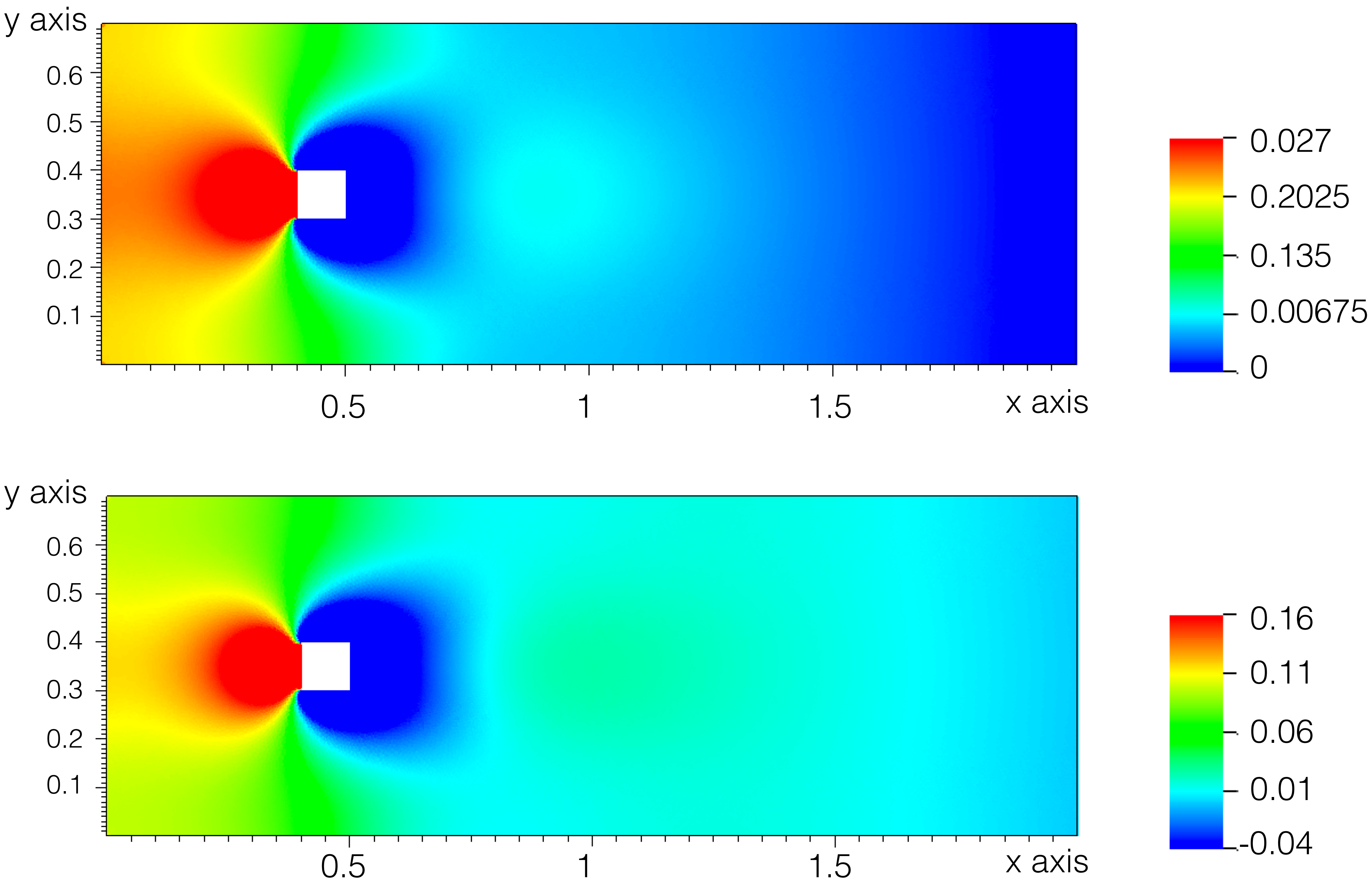
Steady test case

Steady case : x-component of the velocity and its sensitivity



Steady test case

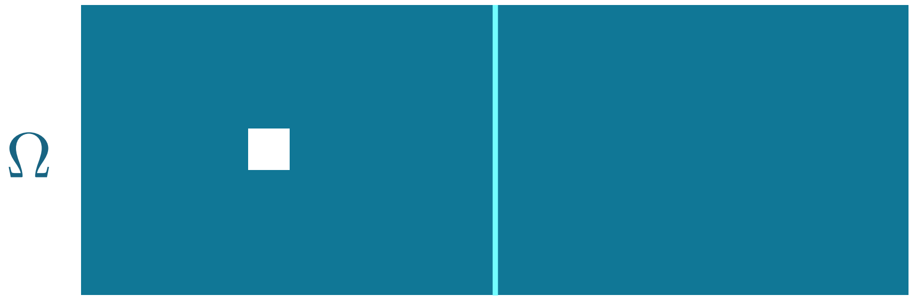
Steady case : pressure and its sensitivity


State

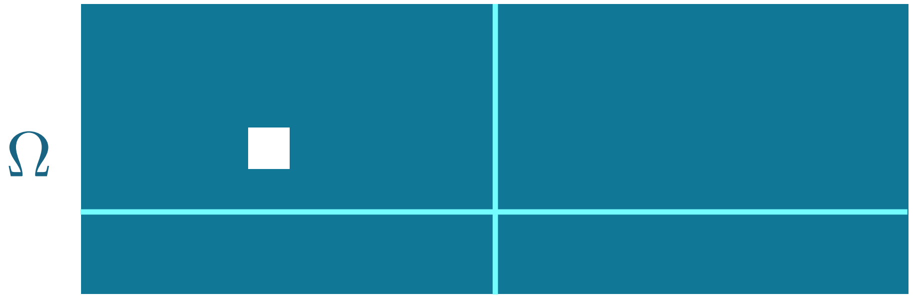
Sensitivity

Steady test case

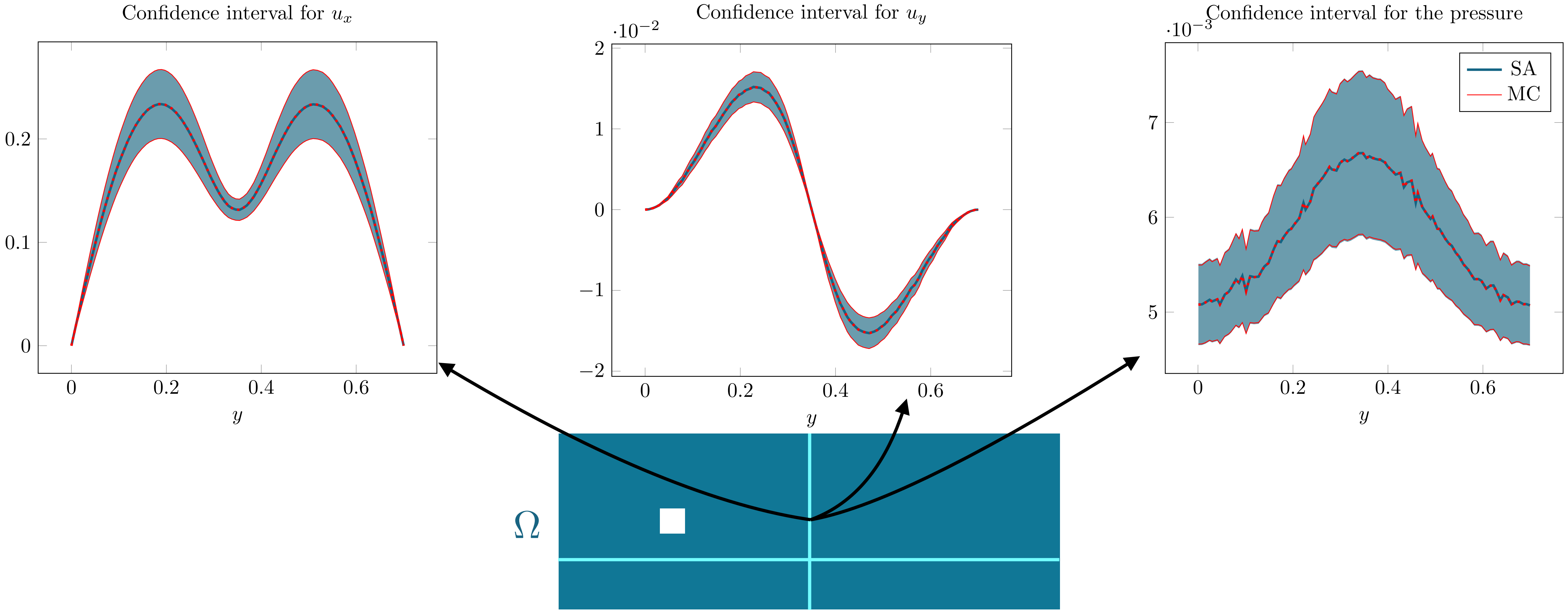
Steady test case



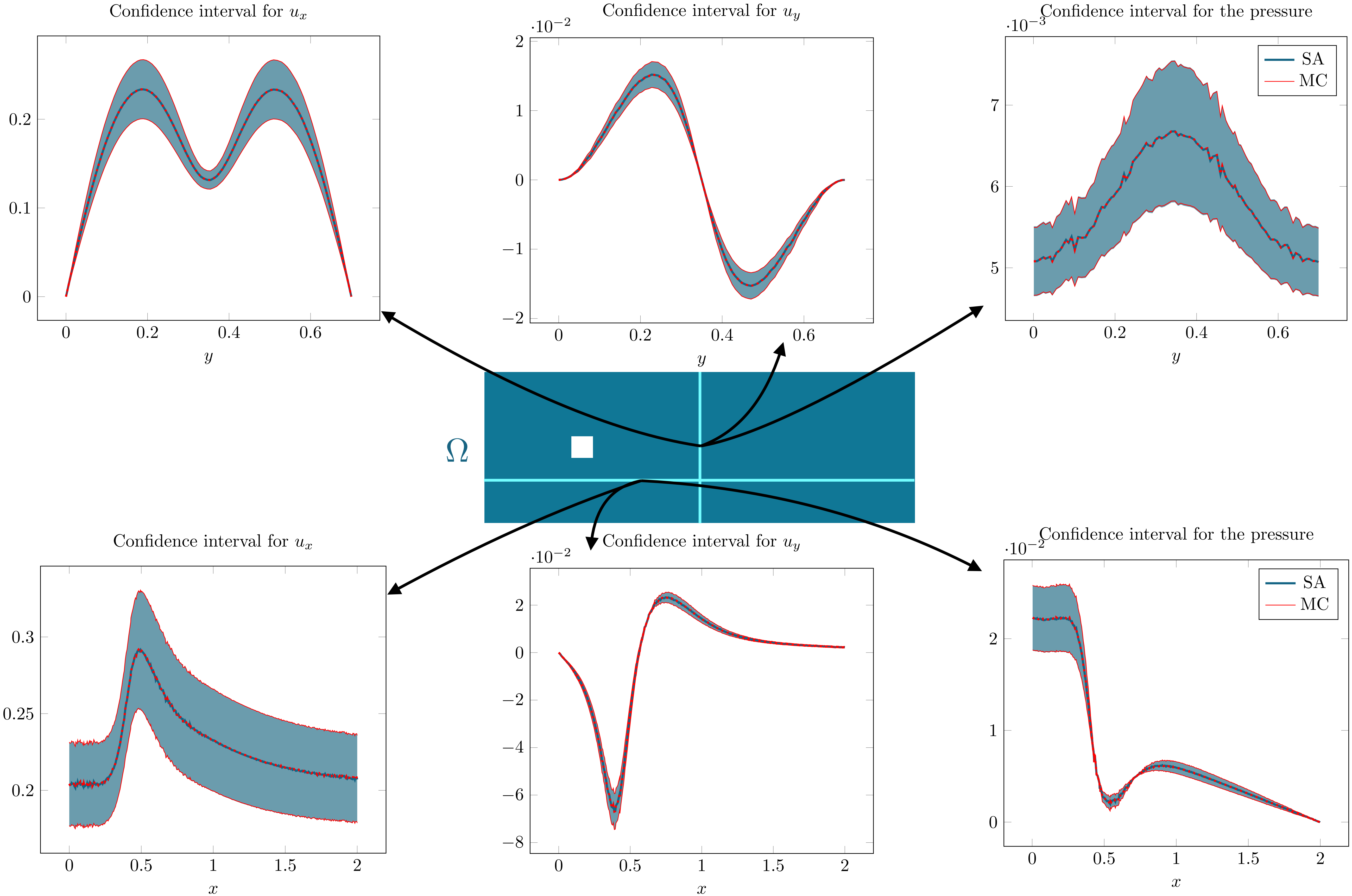
Steady test case



Steady test case

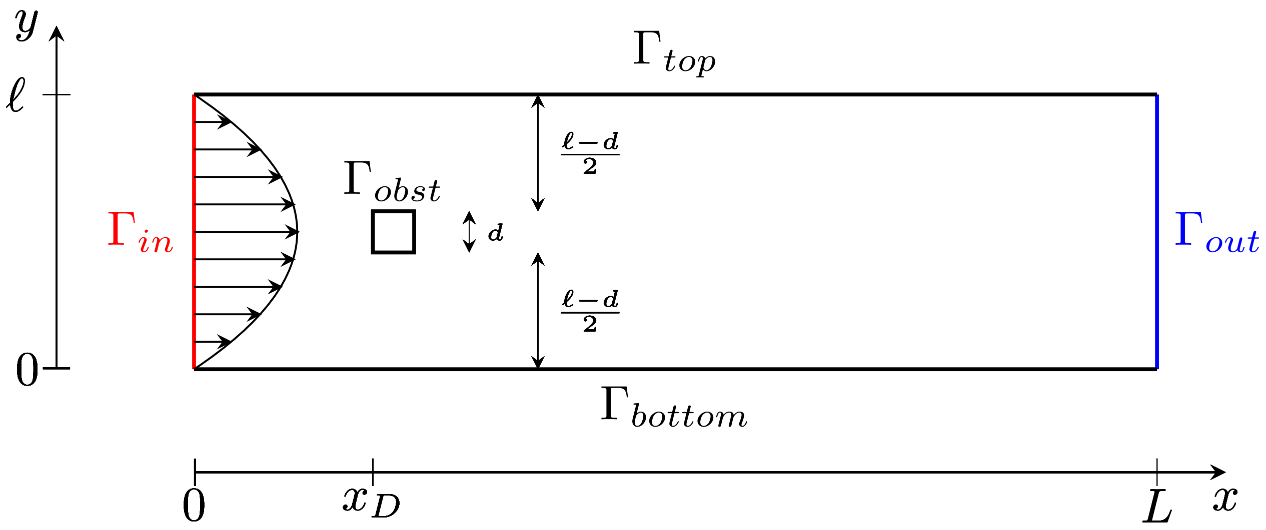


Steady test case



Unsteady test case

Domain :



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Unsteady test case

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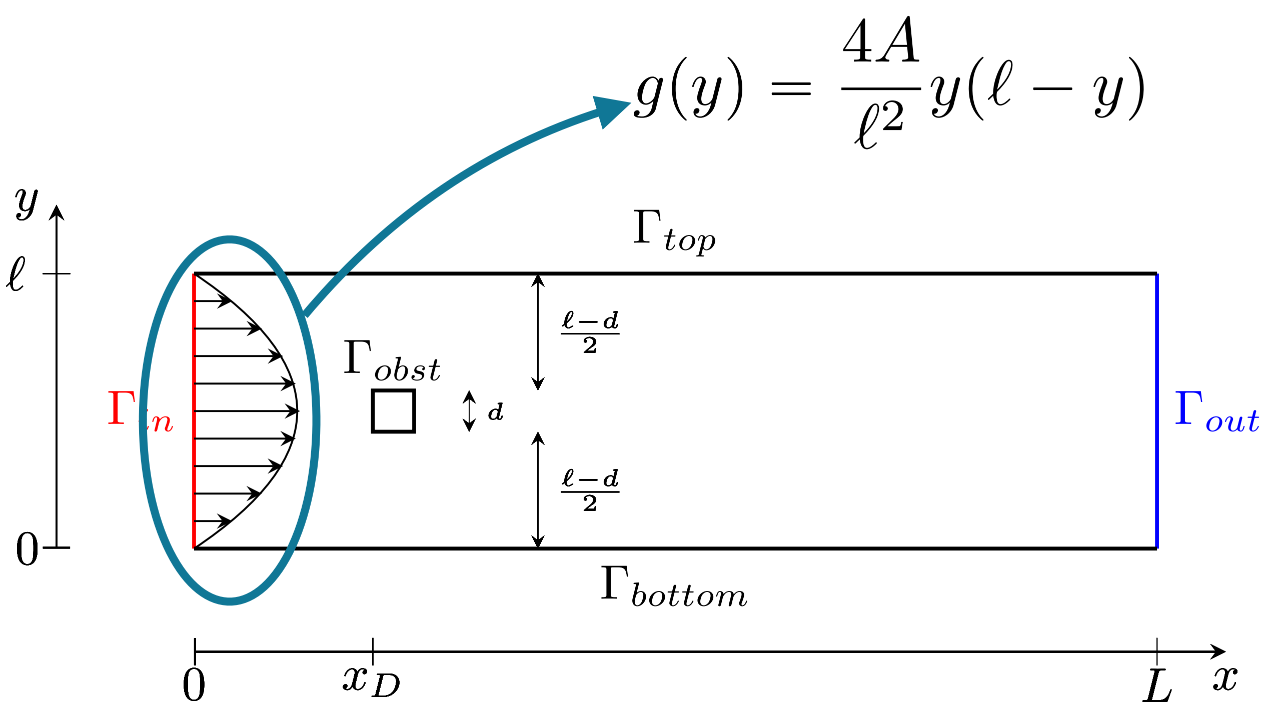

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Unsteady test case

Domain :

uncertain parameter

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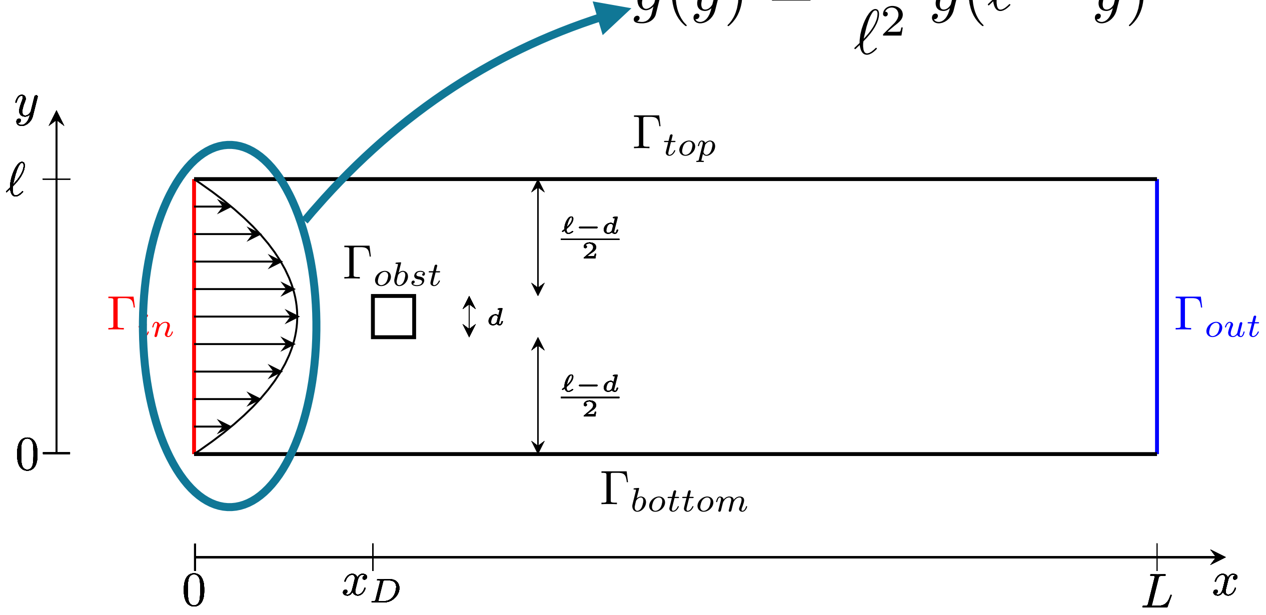

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Unsteady test case

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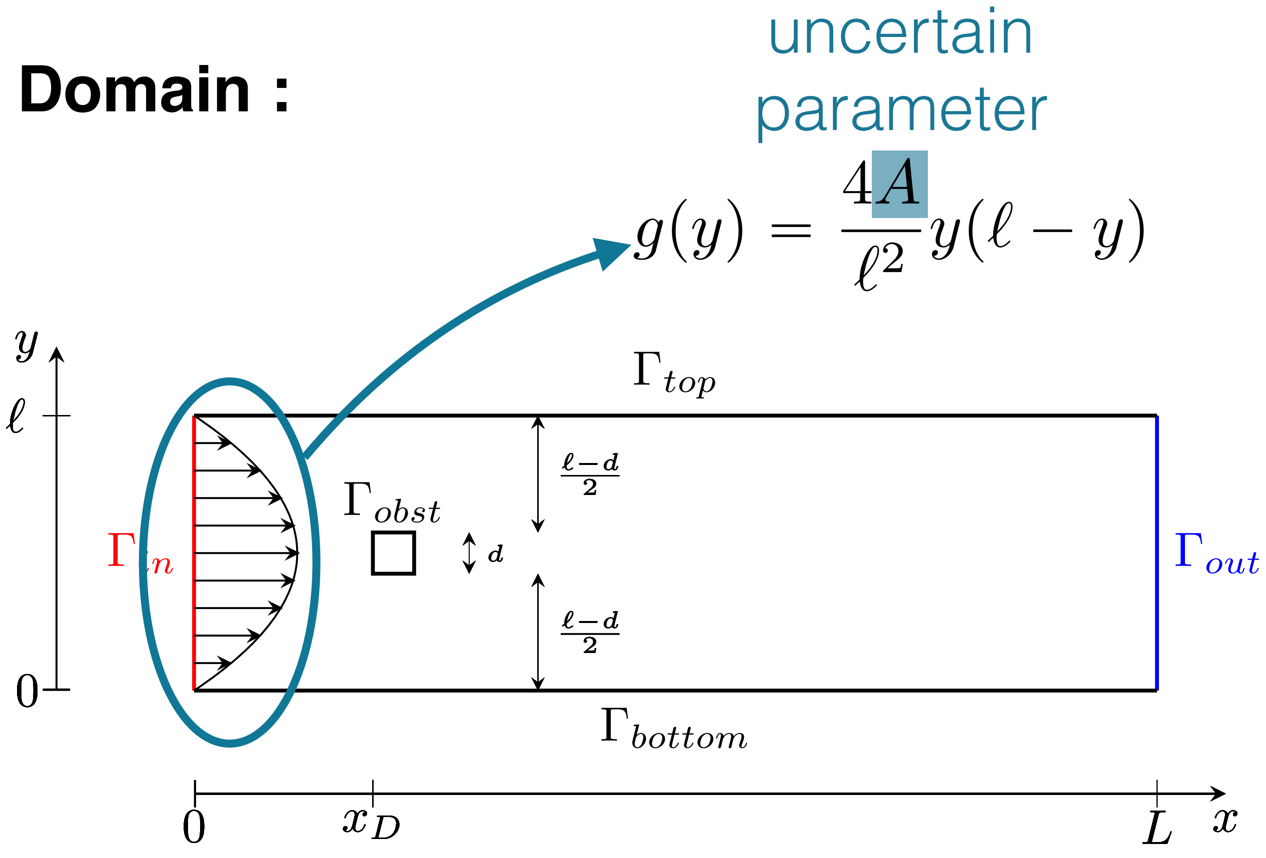

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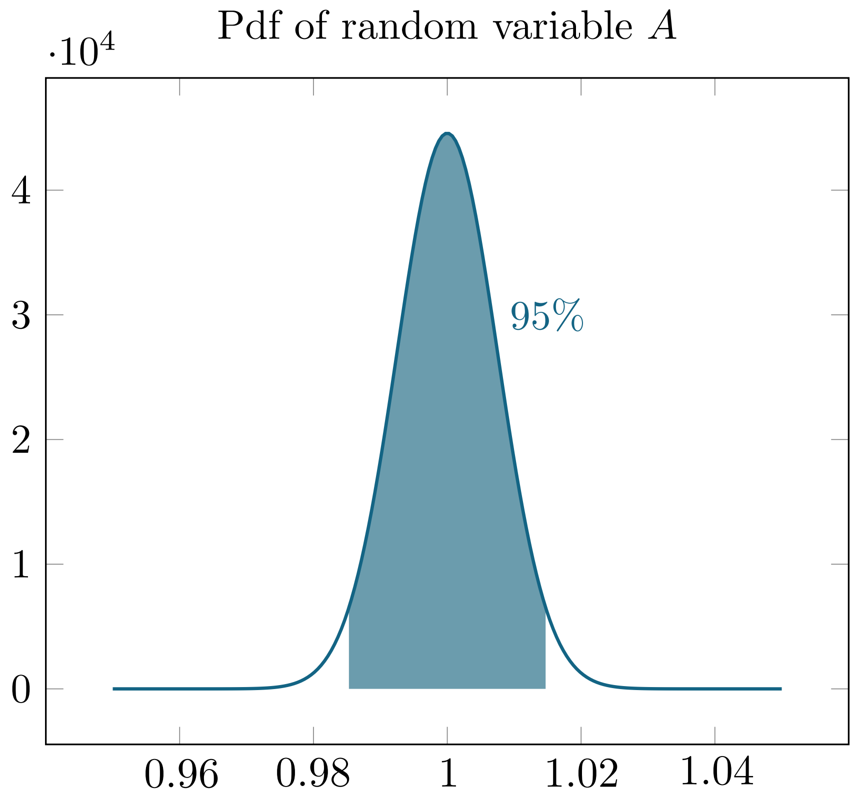
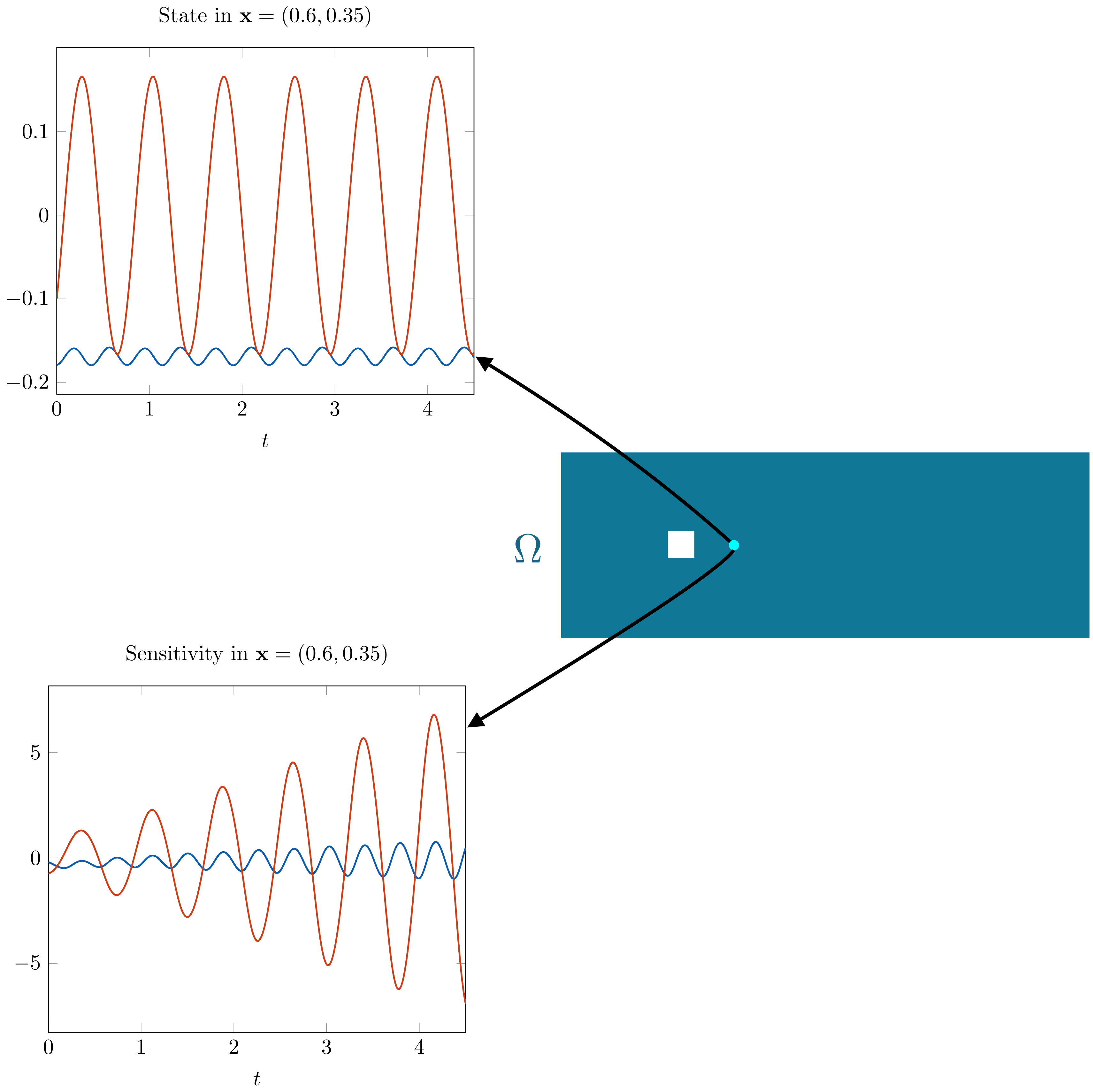


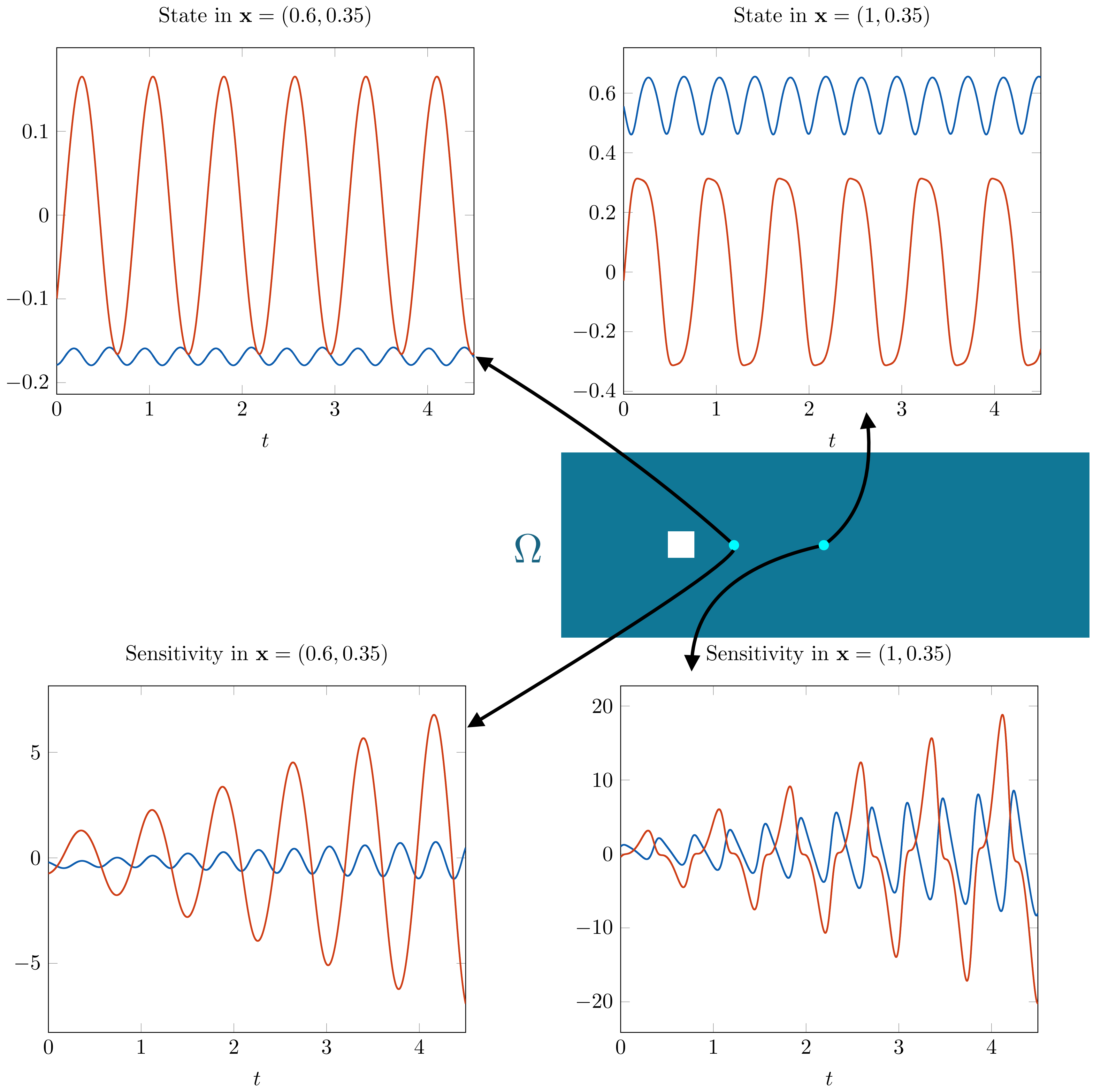
Numerical results

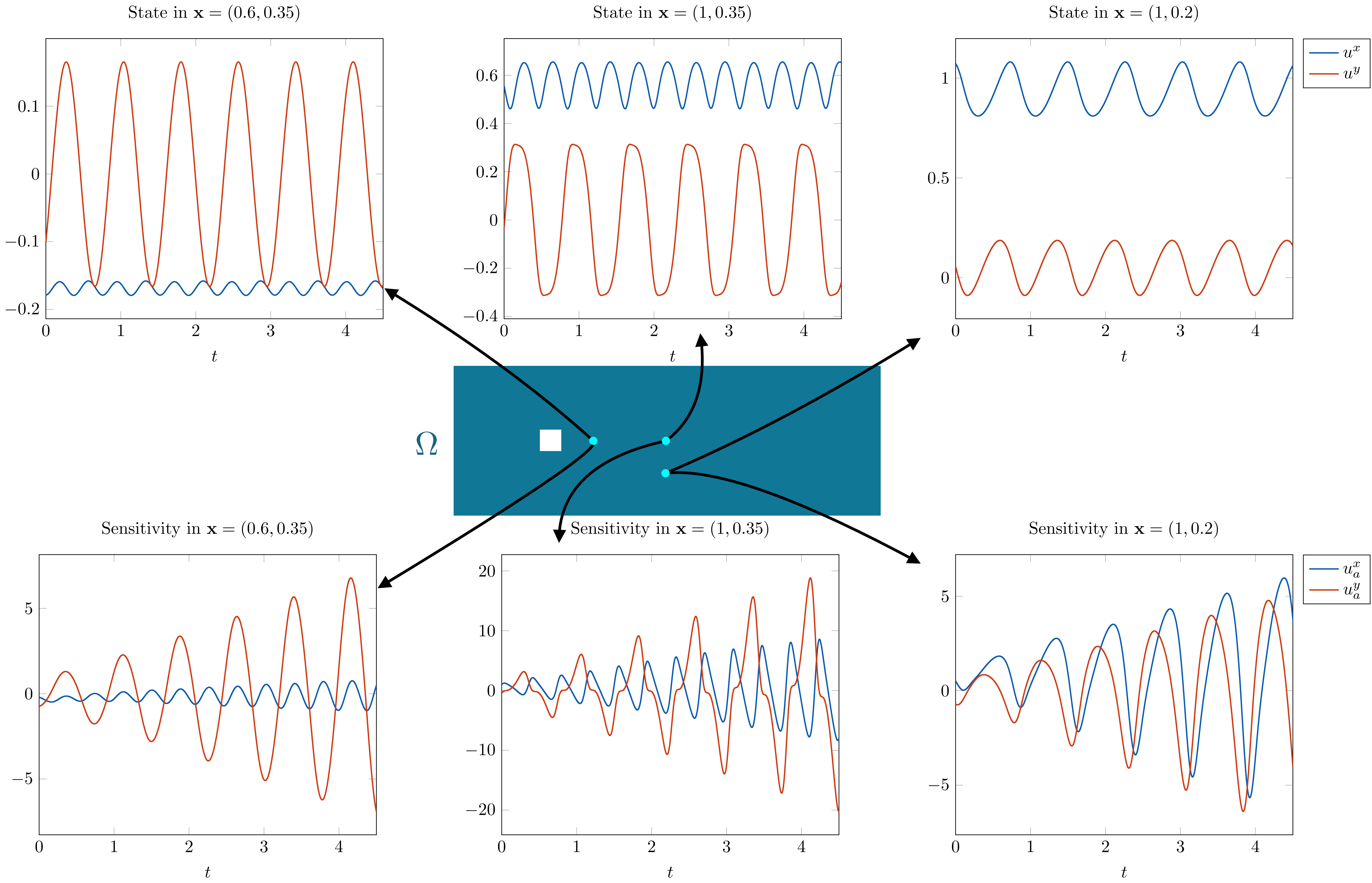


State in $\mathbf{x} = (0.6, 0.35)$

Sensitivity in $\mathbf{x} = (0.6, 0.35)$

Numerical results




State in $\mathbf{x} = (0.6, 0.35)$
State in $\mathbf{x} = (1, 0.35)$
State in $\mathbf{x} = (1, 0.2)$
u^x
u^y
Ω
Sensitivity in $\mathbf{x} = (0.6, 0.35)$
Sensitivity in $\mathbf{x} = (1, 0.35)$
Sensitivity in $\mathbf{x} = (1, 0.2)$
u_a^x
u_a^y
t

Filtering procedure

It is reasonable to assume that the velocity behaves as follows:

$$\mathbf{u}(\mathbf{x}, t; a) \simeq \sum_{k=0}^{N} \mathbf{u}_{0,k}(\mathbf{x}; a) \cos(\omega_k(a)t).$$

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By differentiating this with respect to the parameter of interest, one obtains the following behaviour for the sensitivity:

$$\mathbf{u}_a(\mathbf{x}, t; a) = \sum_{k=0}^{N} \mathbf{u}_{0,a,k}(\mathbf{x}; a) \cos(\omega_k(a)t) - t \sum_{k=0}^{N} \mathbf{u}_{0,k}(\mathbf{x}; a)\omega_k'(a) \sin(\omega_k(a)t)$$

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Bounded **Unbounded**

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We propose a filter to recover the first part of the sensitivity.

Filtering procedure

We propose a filter to recover the bounded part of the sensitivity

$$\mathbf{u}_a(\mathbf{x}, t; a) = \sum_{k=0}^{N} \mathbf{u}_{0,a,k}(\mathbf{x}; a) \cos(\omega_k(a)t) - t \sum_{k=0}^{N} \mathbf{u}_{0,k}(\mathbf{x}; a)\omega_k'(a) \sin(\omega_k(a)t)$$

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$$\mathbf{u}_a(\mathbf{x},t;a) = \underbrace{\boxed{\sum_{k=0}^{N} \mathbf{u}_{0,a,k}(\mathbf{x};a)\cos(\omega_k(a)t)}}_{=:\,\mathbf{h}(\mathbf{x},t;a)} - t\sum_{k=0}^{N} \mathbf{u}_{0,k}(\mathbf{x};a)\omega_k'(a)\sin(\omega_k(a)t)$$

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One can identify the period T of the state using the Fourier transform, and therefore $\forall t$ the following equality holds:

$$\int_{t-T}^{t} \frac{\mathbf{u}_a(\mathbf{x}, s; a)}{s} ds = \int_{t-T}^{t} \frac{\mathbf{h}(\mathbf{x}, s; a)}{s} ds =: \mathcal{I}(t)$$

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The left-hand side can be computed numerically. Finite differences can be used to numerically compute the following derivative:

$$\frac{d\mathcal{I}}{dt} = \frac{\mathbf{h}(\mathbf{x},t;a)}{t} - \frac{\mathbf{h}(\mathbf{x},t;a)}{T-t}$$

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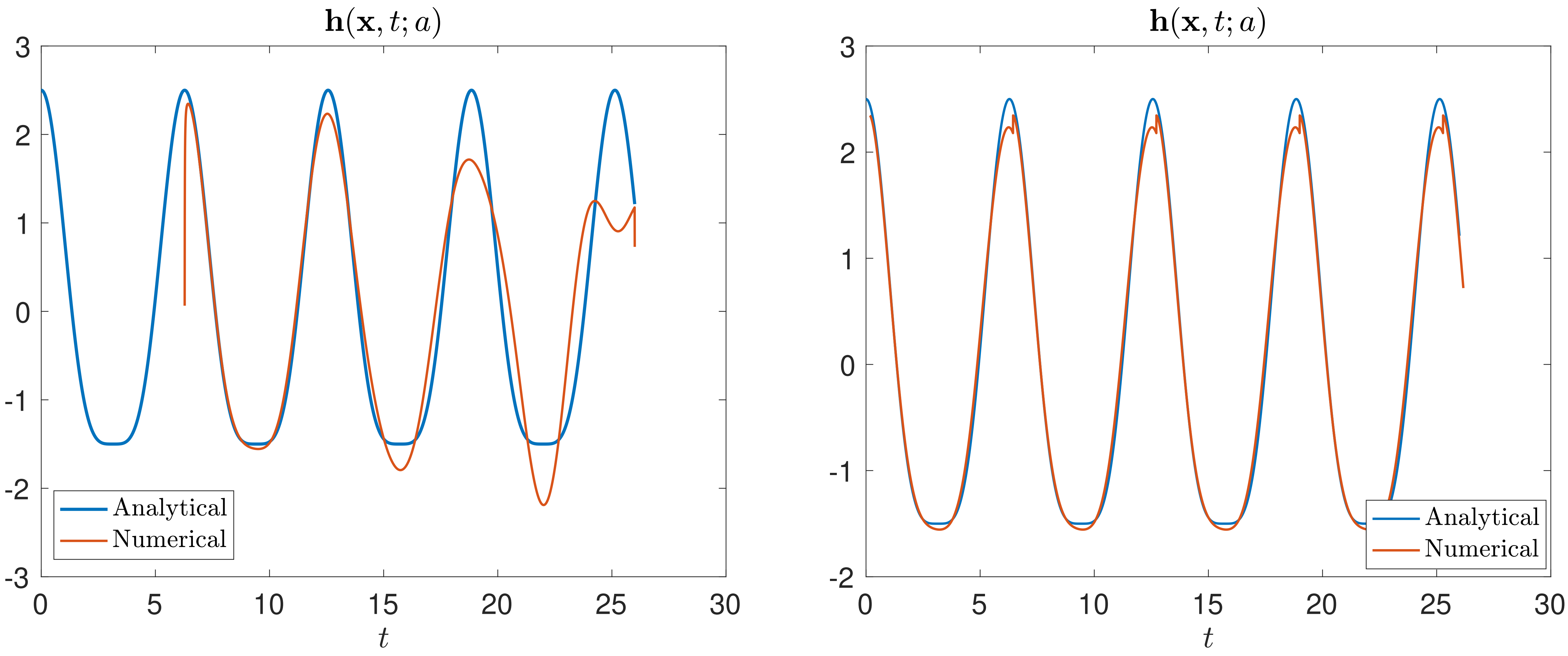
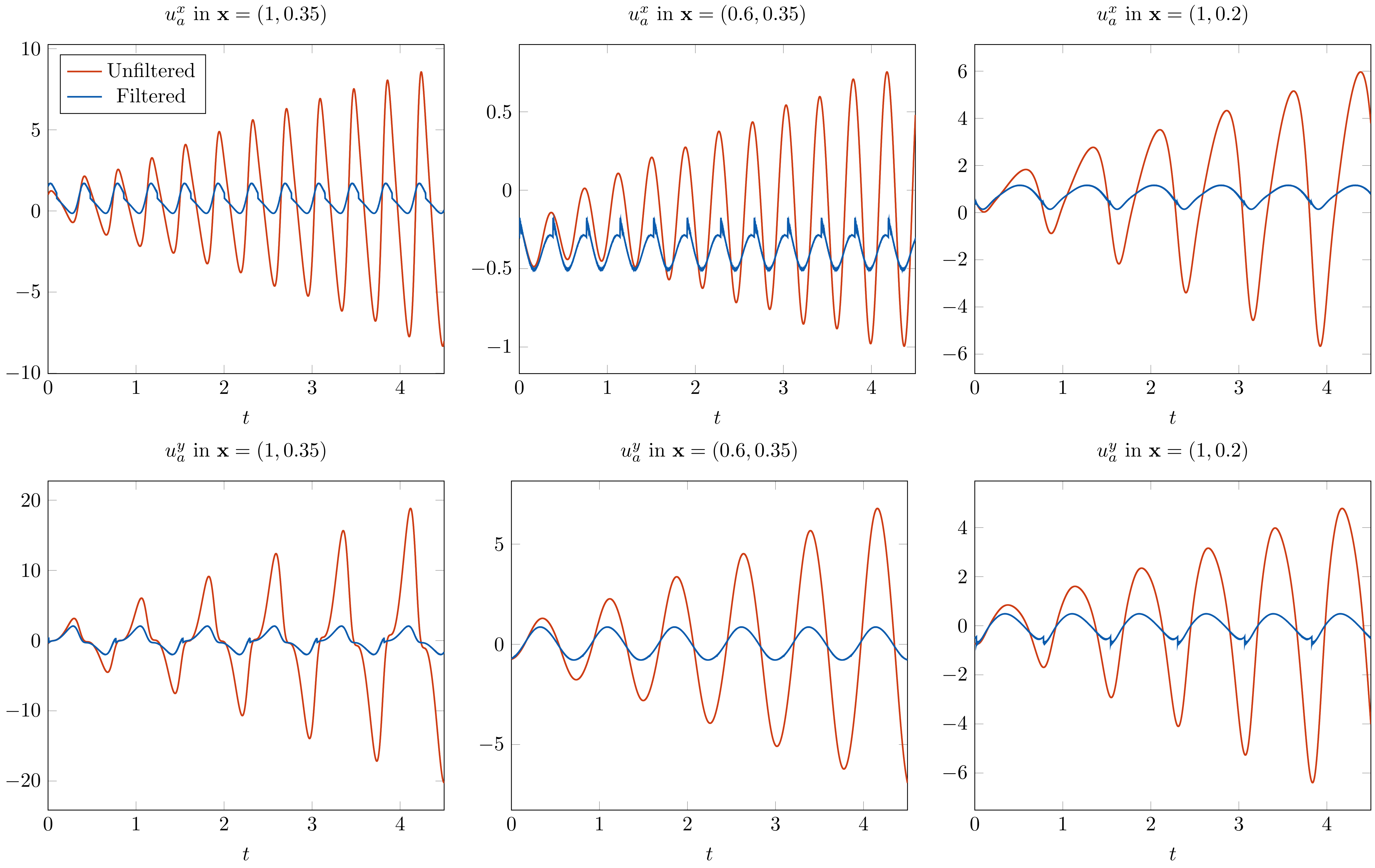
Finally, one has:

$$\mathbf{h}(\mathbf{x},t;a) = \frac{t(T-t)}{T}\frac{d\mathcal{I}}{dt}$$

We verify the filter on an analytical case



Filtering procedure



u_a^x in $\mathbf{x} = (1, 0.35)$

u_a^x in $\mathbf{x} = (0.6, 0.35)$

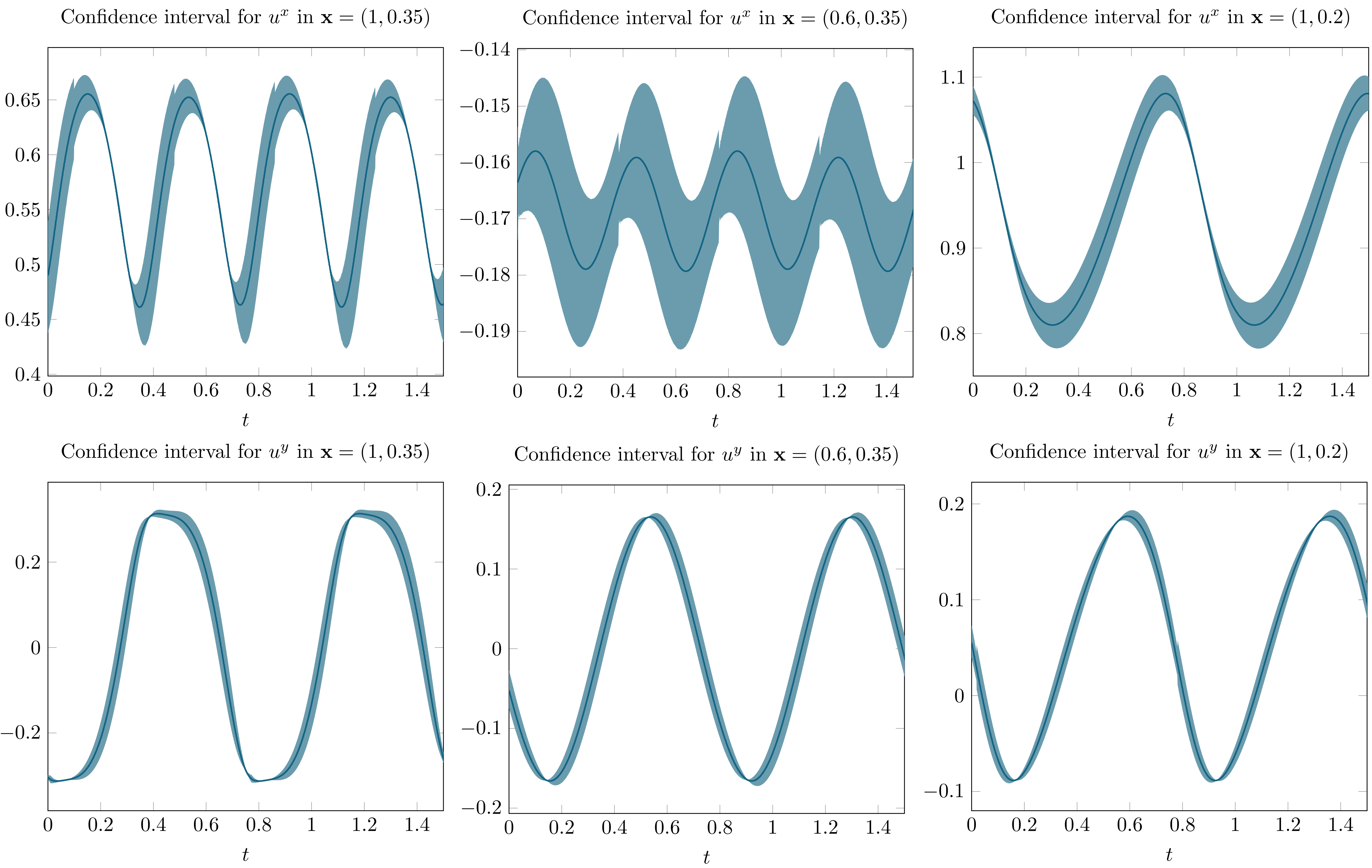
u_a^x in $\mathbf{x} = (1, 0.2)$

u_a^y in $\mathbf{x} = (1, 0.35)$

u_a^y in $\mathbf{x} = (0.6, 0.35)$

u_a^y in $\mathbf{x} = (1, 0.2)$

Confidence intervals with filtered sensitivities



Confidence interval for u^x in $\mathbf{x} = (1, 0.35)$

Confidence interval for u^x in $\mathbf{x} = (0.6, 0.35)$

Confidence interval for u^x in $\mathbf{x} = (1, 0.2)$

Confidence interval for u^y in $\mathbf{x} = (1, 0.35)$

Confidence interval for u^y in $\mathbf{x} = (0.6, 0.35)$

Confidence interval for u^y in $\mathbf{x} = (1, 0.2)$

Conclusion and perspectives

Conclusion:

- We implemented an efficient UQ tool
- The comparison to Monte Carlo in the steady case validates the approach
- A filtering procedure for the unsteady case is necessary

Perspectives:

- Extension to 3D
- Coupling with a temperature equation
- Turbulence models
- More realistic simulations
- Accounting for the variance in frequency

Thank you

for your attention!

Lift function

We build a vector field $\mathbf{R}_g(x,y) = (R_g^x(x,y), R_g^y(x,y))^t$ which is solenoidal, smooth enough and equal to $\mathbf{u}$ on the Dirichlet boundary.

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To do this, we need two functions, $g_F(y)$ and $\mathcal{H}(x)$, with the following properties:

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$$\begin{aligned}
g_F(0) = g_F(\ell) = 0 \\
g_F(y) = 0 \ \forall y \in \left[\frac{\ell - d}{2}, \frac{\ell + d}{2}\right], \\
g_F \in C^0[0,\ell], \quad \partial_y g_F \in L^\infty[0,\ell] \\
\int_0^\ell g_F(y)dy = \int_0^\ell g(y)dy.
\end{aligned}$$

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$$\begin{aligned} \mathcal{H}(0) &= 1, \\ \mathcal{H}(x) &= 0 \ \forall x \in [x_D, L], \\ \mathcal{H}'(0) &= \mathcal{H}'(x_D) = 0 \\ \mathcal{H} &\in C^2[0,L] \end{aligned}$$

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Then, the following vector field has all the required properties:

$$R_g^x(x, y) = g(y)\mathcal{H}(x) + g_F(y)(1 - \mathcal{H}(x)) \qquad R_g^y(x, y) = -\mathcal{H}'(x)(G(y) - G_F(y))$$

Lift function