Sensitivity analysis for the Euler equations in Lagrangian coordinates

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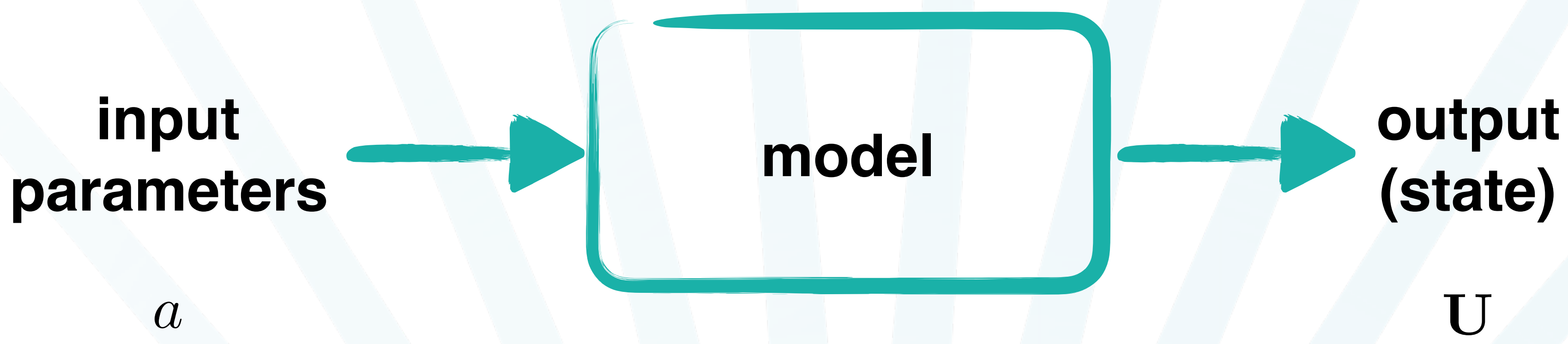
Outline of the talk

- Sensitivity analysis
- Sensitivity analysis for hyperbolic equations
- Euler equations in barotropic conditions (p-system)
- Classical numerical schemes and results
- Anti-diffusive numerical scheme and results

Sensitivity Analysis

Sensitivity Analysis

Sensitivity analysis: study of how changes in the **inputs** of a model affect the **outputs**



Sensitivity: $$\frac{\partial \mathbf{U}}{\partial a} = \mathbf{U}_a$$

- Optimization

Applications

- Optimization

Problem: $\min_{a \in \mathcal{A}} J(\mathbf{U})$, where $J(\mathbf{U}) = \frac{1}{2} b(\mathbf{U}, \mathbf{U})$ and b is bilinear.

Classical optimization techniques call for the differentiation of the cost function:

$$\frac{\partial J(\mathbf{U})}{\partial a} = b(\mathbf{U}, \mathbf{U}_a)$$

Applications

- Optimization
- Quick evaluation of close solutions

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$$\mathbf{U}(a+\delta a) = \mathbf{U}(a) + \delta a \mathbf{U}_a(a) + o(\delta a^2)$$

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- Uncertainty quantification

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First order estimates

$$\mu \qquad \mathbf{U}(\mu_a)$$

$$\sigma^2 \qquad \mathbf{U}_a(\mu_a)^T \mathbf{U}_a(\mu_a)\sigma_a^2$$

Standard techniques of sensitivity analysis

Standard techniques of sensitivity analysis call for the differentiation of the state system:

$$\begin{cases} \partial_t \mathbf{U} + \nabla \mathbf{F}(\mathbf{U}) = 0 & \Omega \times (0,T), \\ \mathbf{U}(x,0) = \mathbf{g}(x) & \Omega, \end{cases}$$

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If these techniques are applied to hyperbolic equations, **Dirac delta functions** will appear in the sensitivity.

Sensitivity analysis for hyperbolic equations

Definition of the source term

In order to have a sensitivity system which is valid also when the state is discontinuous, we add a correction term:

$$\begin{cases} \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S} & \Omega \times (0,T), \\ \mathbf{U}_a(x,0) = \mathbf{g}_a(x) & \Omega, \end{cases}$$

defined as follows:

$$\mathbf{S} = \sum_{k=1}^{N_s} \boldsymbol{\rho}_k \delta(x - x_{k,s}(t)),$$

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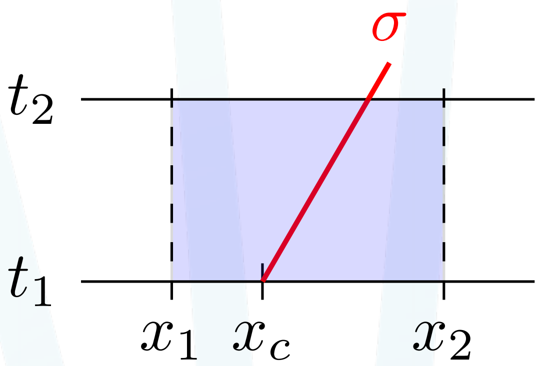
$\boldsymbol{\rho}_k$: amplitude of the k-th correction (to be computed)

Definition of the source term

To compute the amplitude of the correction, we consider an infinitesimal control volume containing a single discontinuity:



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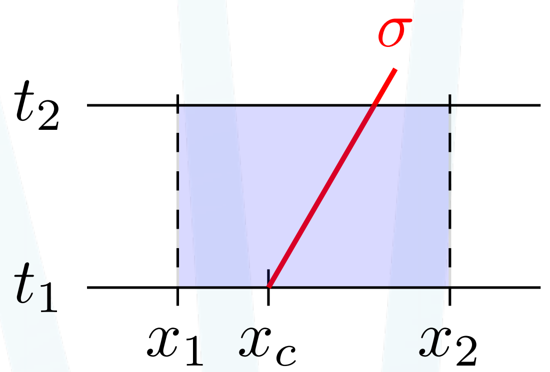

By integrating the sensitivity equations with the source term on the control volume, one has:

$$\boldsymbol{\rho} = (\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma + \mathbf{F}_a^+ - \mathbf{F}_a^-$$

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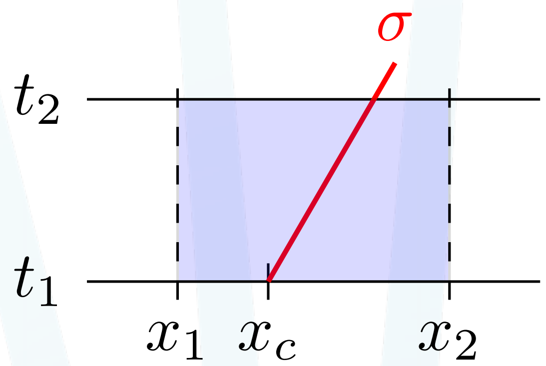

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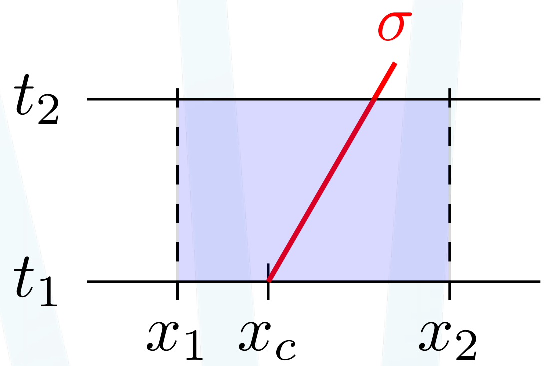

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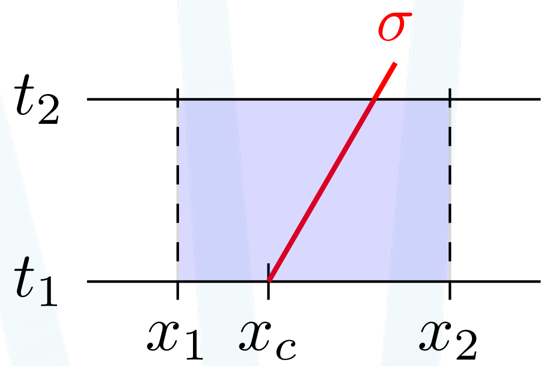

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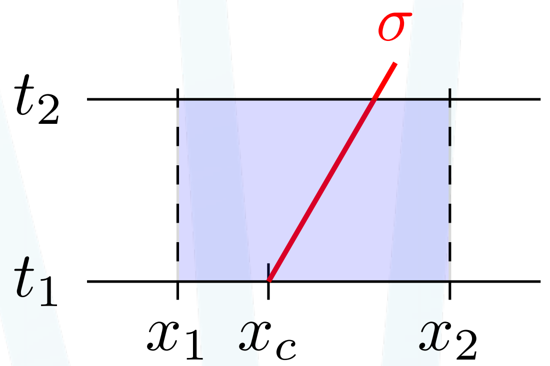

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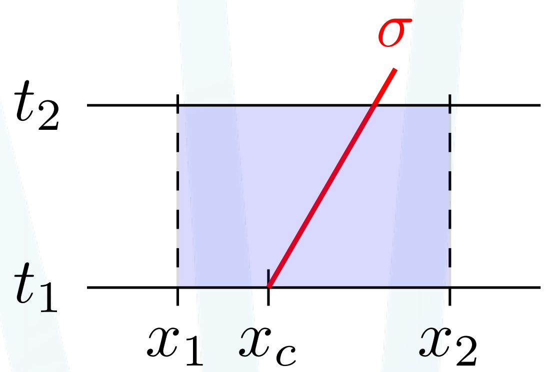

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A shock detector is necessary

Euler equations in barotropic conditions: the *p*-system

The Riemann problem for the *p*-system

The *p*-system writes:

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = 0. \end{cases}$$

The Riemann problem for the *p*-system

The *p*-system writes:

$$\begin{cases} \partial_t \tau - \partial_x u = 0, & p'(\tau) < 0 \\ \partial_t u + \partial_x p(\tau) = 0. & p''(\tau) > 0 \end{cases}$$

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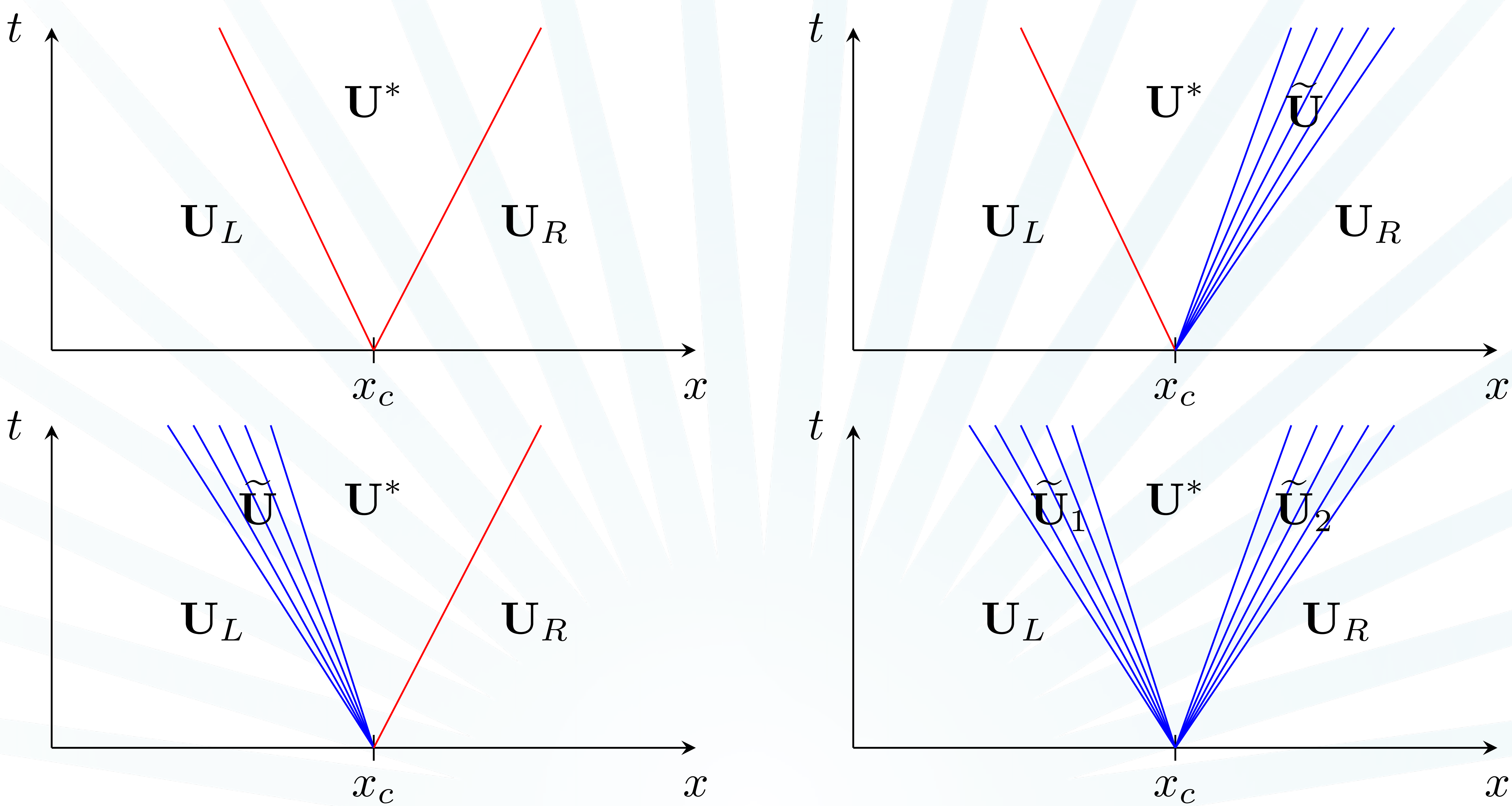

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$$g_1(\tau; \mathbf{U}_L) = \begin{cases} u_L - \sqrt{-(\tau^{-\gamma} - \tau_L^{-\gamma})(\tau - \tau_L)} & \text{if } \tau \leq \tau_L, \\ u_L + \frac{2\sqrt{\gamma}}{1-\gamma}(\tau^{\frac{1-\gamma}{2}} - \tau_L^{\frac{1-\gamma}{2}}) & \text{if } \tau > \tau_L. \end{cases}$$

$$g_2(\tau; \mathbf{U}_R) = \begin{cases} u_R + \sqrt{-(\tau^{-\gamma} - \tau_R^{-\gamma})(\tau - \tau_R)} & \text{if } \tau \leq \tau_R, \\ u_R + \frac{2\sqrt{\gamma}}{1-\gamma}(\tau_R^{\frac{1-\gamma}{2}} - \tau^{\frac{1-\gamma}{2}}) & \text{if } \tau > \tau_R. \end{cases}$$

$$u^* = g_1(\tau^*; \mathbf{U}_L) = g_2(\tau^*; \mathbf{U}_R)$$

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$$\tilde{\tau}_1 = \left(\frac{\gamma t^2}{(x - x_c)^2} \right)^{\frac{1}{\gamma+1}} \qquad \tilde{u}_1 = u_L + \frac{2\sqrt{\gamma}}{1-\gamma} \left(\tilde{\tau}_1^{\frac{1-\gamma}{2}} - \tau_L^{\frac{1-\gamma}{2}} \right)$$

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$$u_a^* = g_1'(\tau^*; \mathbf{U}_L)\tau_a^* + \frac{\partial g_1}{\partial \tau_L}(\tau^*; \mathbf{U}_L)\tau_{a,L} + \frac{\partial g_1}{\partial u_L}(\tau^*; \mathbf{U}_L)u_{a,L} =$$

$$= g_2'(\tau^*; \mathbf{U}_R)\tau_a^* + \frac{\partial g_2}{\partial \tau_R}(\tau^*; \mathbf{U}_R)\tau_{a,R} + \frac{\partial g_2}{\partial u_R}(\tau^*; \mathbf{U}_R)u_{a,R},$$

$$\tau_a^* = \frac{\frac{\partial g_2}{\partial \tau_R}(\tau^*; \mathbf{U}_R)\tau_{a,R} + \frac{\partial g_2}{\partial u_R}(\tau^*; \mathbf{U}_R)u_{a,R} - \frac{\partial g_1}{\partial \tau_L}(\tau^*; \mathbf{U}_L)\tau_{a,L} - \frac{\partial g_1}{\partial u_L}(\tau^*; \mathbf{U}_L)u_{a,L}}{g_1'(\tau^*; \mathbf{U}_L) - g_2'(\tau^*; \mathbf{U}_R)}$$

$$\tilde{\tau}_{a,1} = 0 \qquad \tilde{u}_{a,1} = \frac{\partial \tilde{u}_1}{\partial a} = u_{a,L} - \sqrt{\gamma}\tau_L^{-\frac{1+\gamma}{2}}\tau_{a,L}$$

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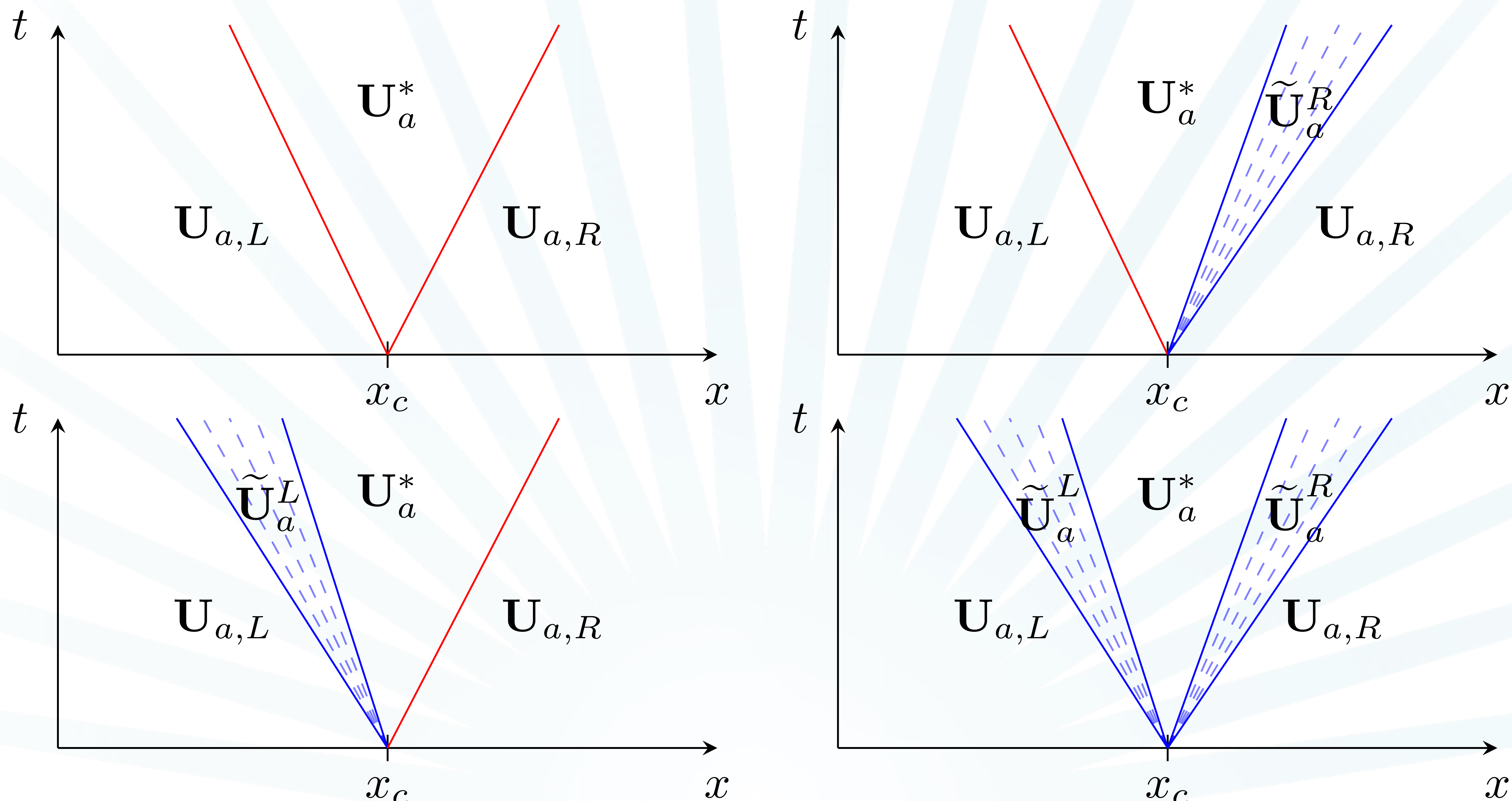

Classical Numerical Schemes

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- Exact Godunov-type scheme

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State:
$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \frac{\Delta t}{\Delta x}(\mathbf{F}(\mathbf{U}^*_{j+1/2}) - \mathbf{F}(\mathbf{U}^*_{j-1/2}))$$

Sensitivity: non conservative, but composed **only** by **discontinuities**

direct average

Classical numerical schemes

- Exact Godunov-type scheme
- First order Roe-type scheme

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State:

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + \frac{\Delta t}{\Delta x}(\lambda_{j-1/2}^{ROE}(\mathbf{U}_{j-1/2}^* - \mathbf{U}_j^n) + \lambda_{j+1/2}^{ROE}(\mathbf{U}_{j+1/2}^* - \mathbf{U}_j^n))$$

Sensitivity: the source term is encompassed in the definition of $\mathbf{U}_{a,j-1/2}^*$:

$$\mathbf{U}_{a,j-1/2}^* = \frac{1}{2}(\mathbf{U}_{a,j-1}^n + \mathbf{U}_{a,j}^n) - \frac{\mathbf{F}_a(\mathbf{U}_j^n, \mathbf{U}_{a,j}^n) - \mathbf{F}_a(\mathbf{U}_{j-1}^n, \mathbf{U}_{a,j}^n)}{2\lambda_{j-1/2}^{ROE}}$$
$$+\frac{\lambda_{a,j-1/2}^{ROE}}{2\lambda_{j-1/2}^{ROE}}\Big((\mathbf{U}_{j-1}^n - \mathbf{U}_{j-1/2}^*)d_{1,j-1} + (\mathbf{U}_j^n - \mathbf{U}_{j-1/2}^*)d_{2,j}\Big).$$

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Shock detectors

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- First order Roe-type scheme
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Time discretisation: two-step Runge-Kutta method

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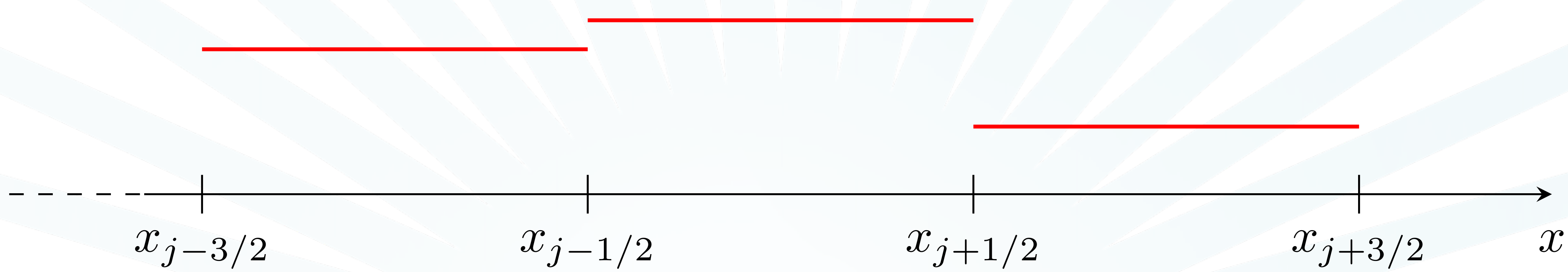

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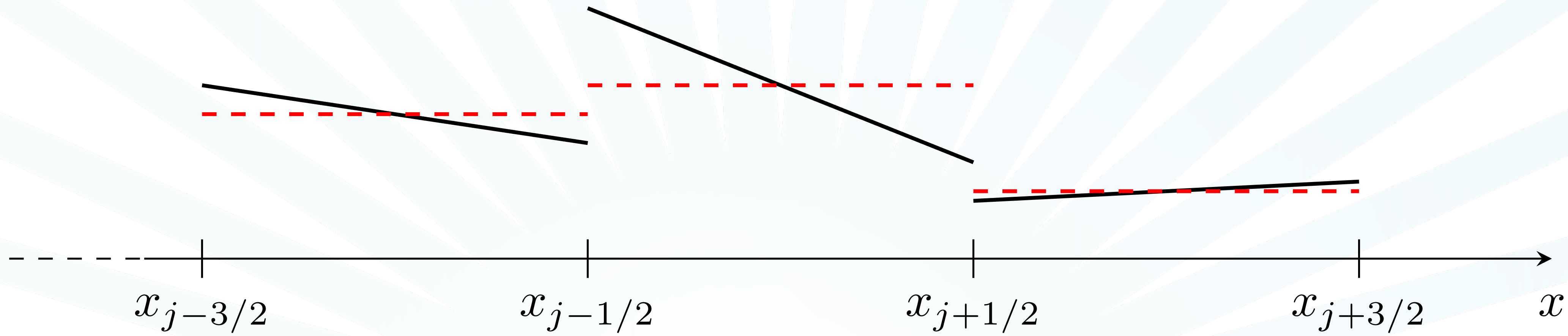
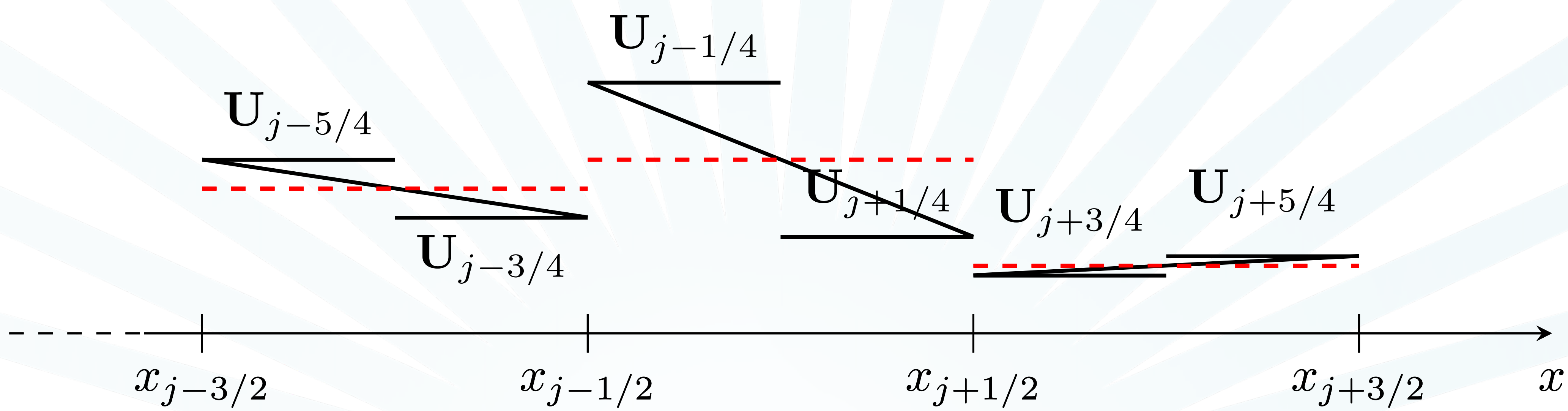


Classical numerical schemes

- Exact Godunov-type scheme
- First order Roe-type scheme
- Second order Roe-type scheme

Time discretisation: two-step Runge-Kutta method

Space discretisation: MUSCL-type scheme



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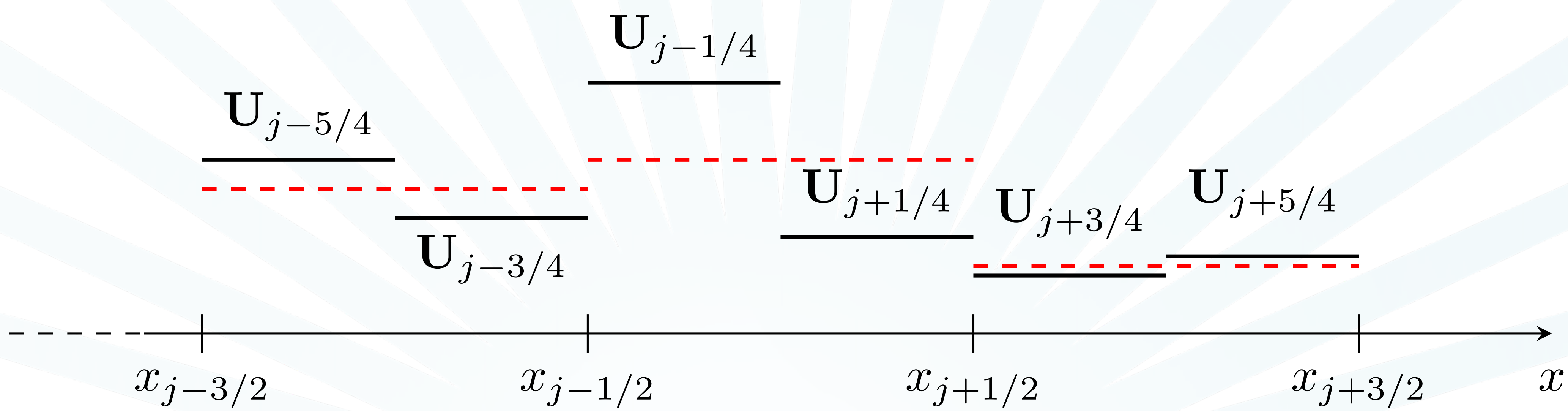
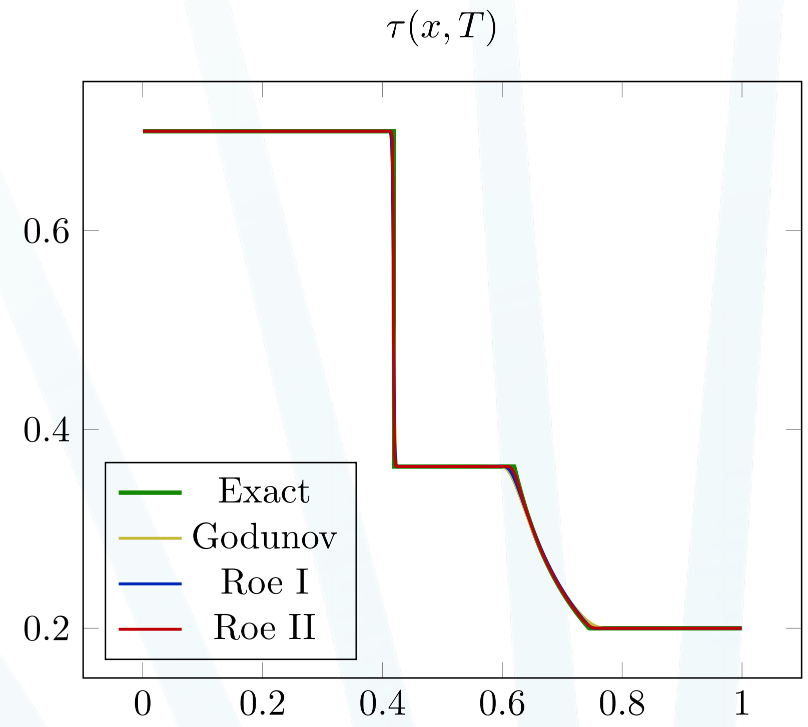
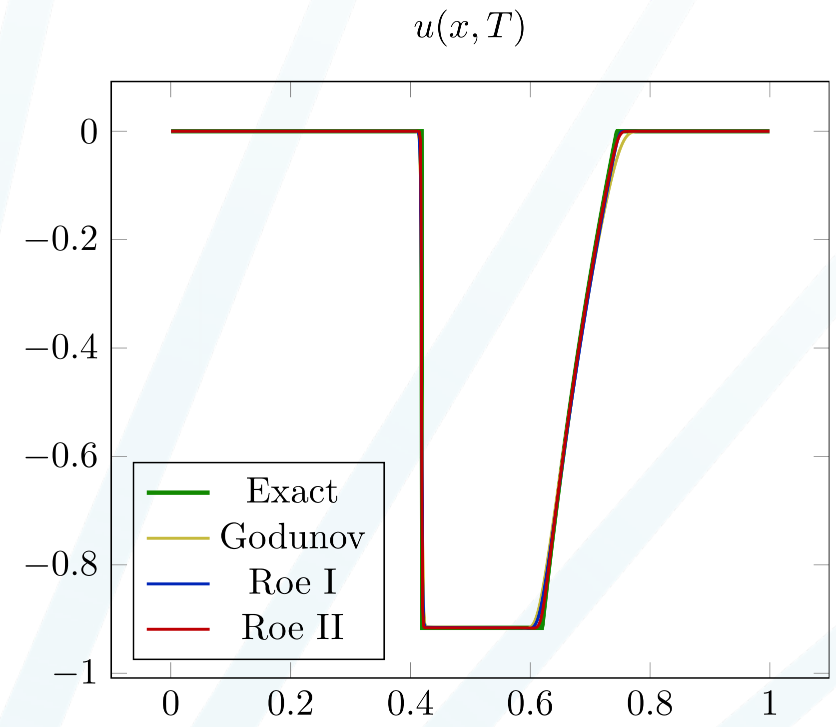


Numerical results

Since the resolution of the state is a classical problem, all the schemes give good results.





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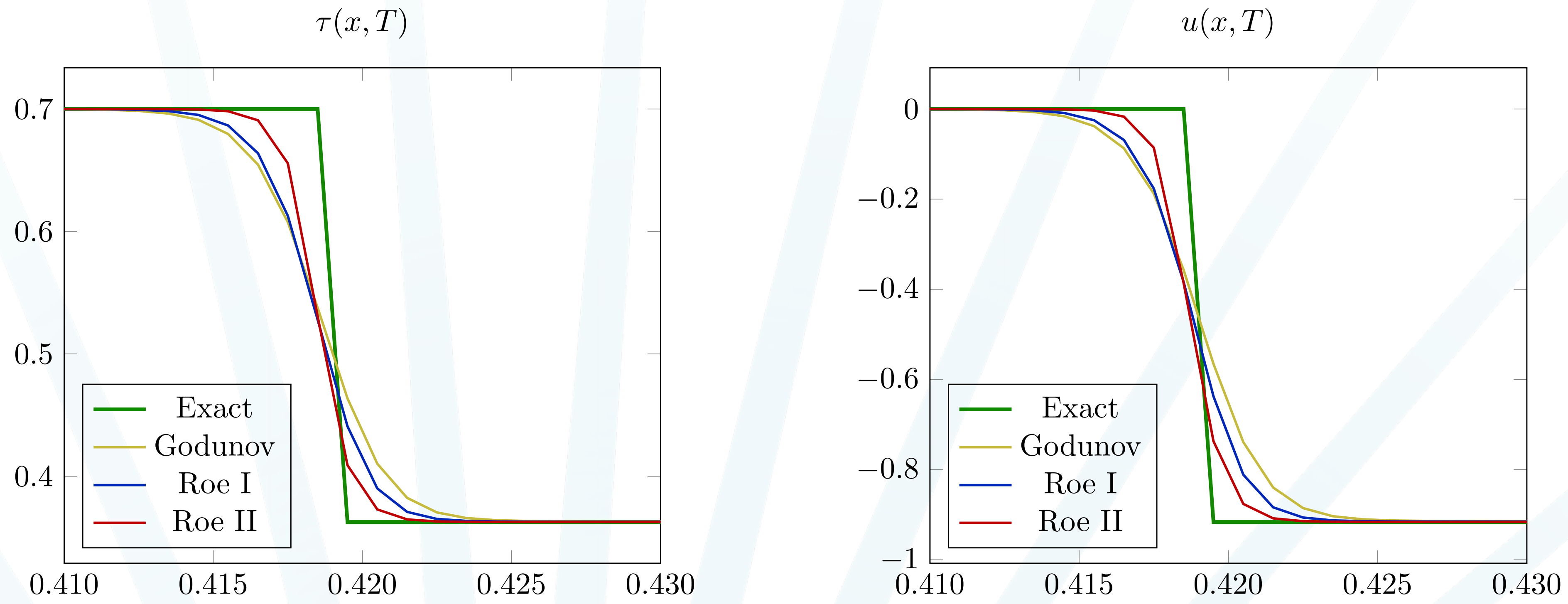

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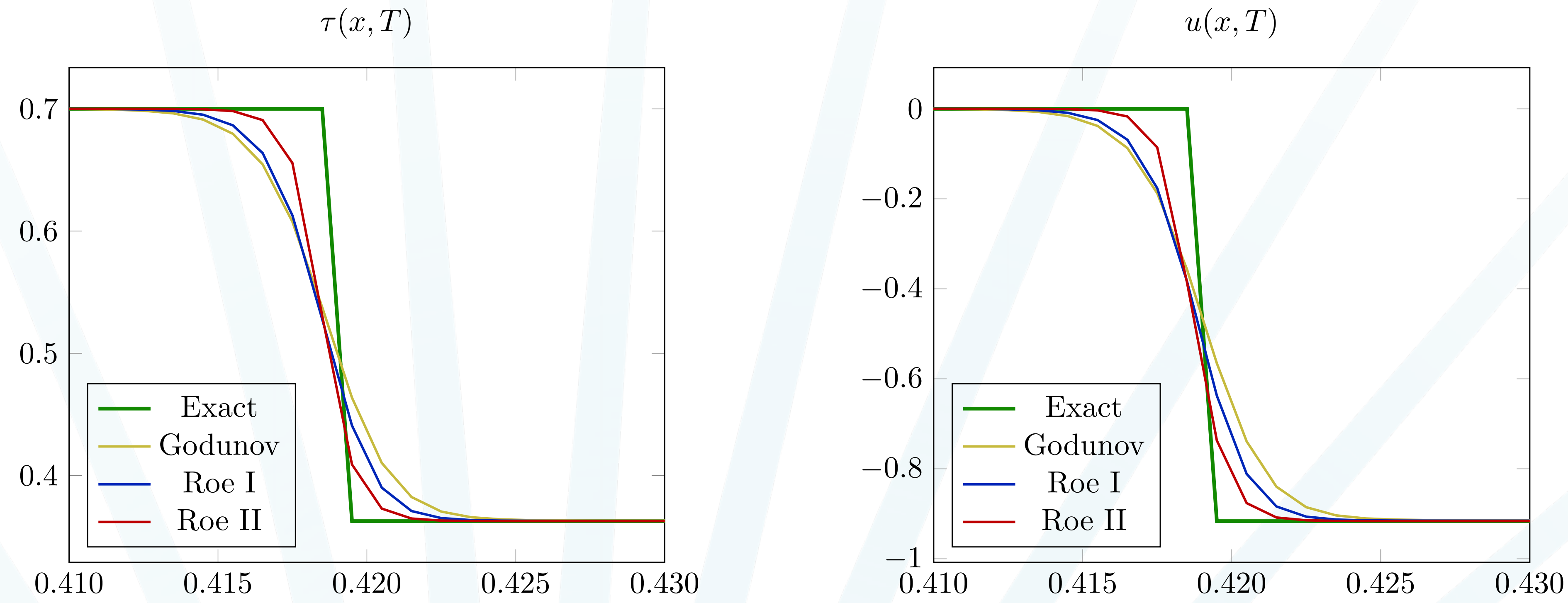
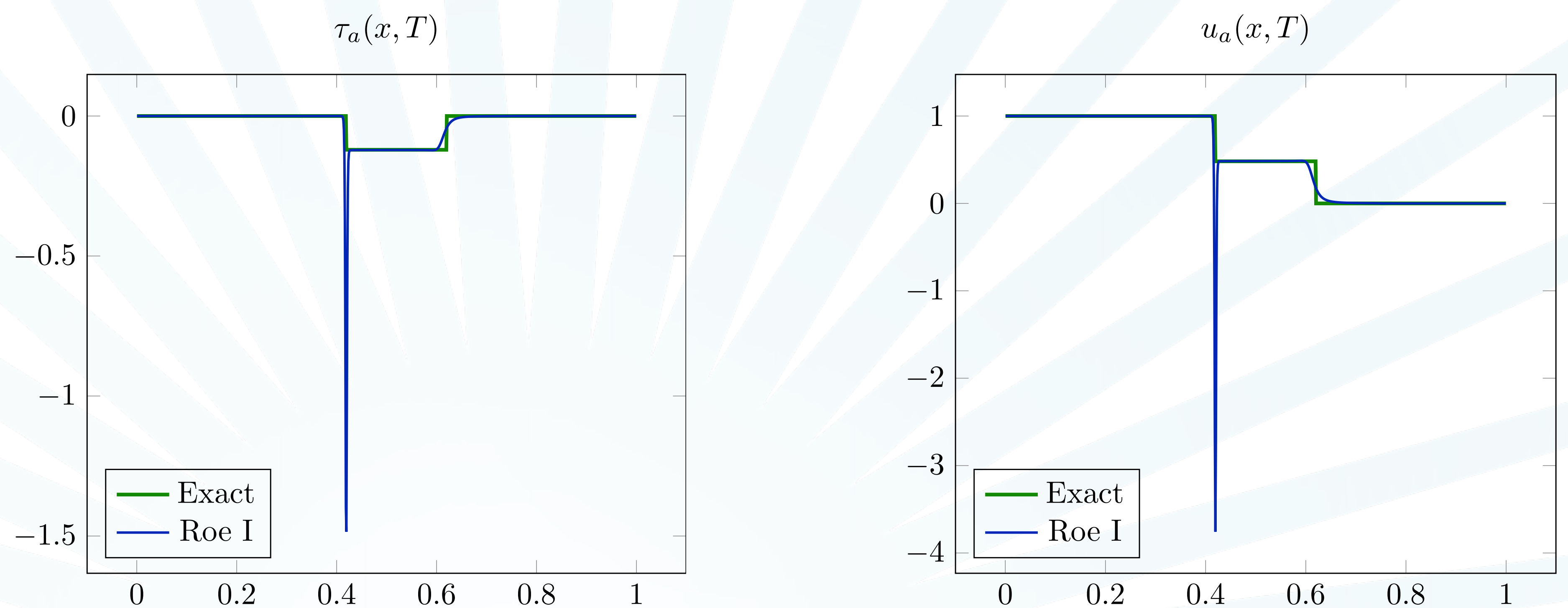


The same schemes do not work **without source term** for the sensitivity.



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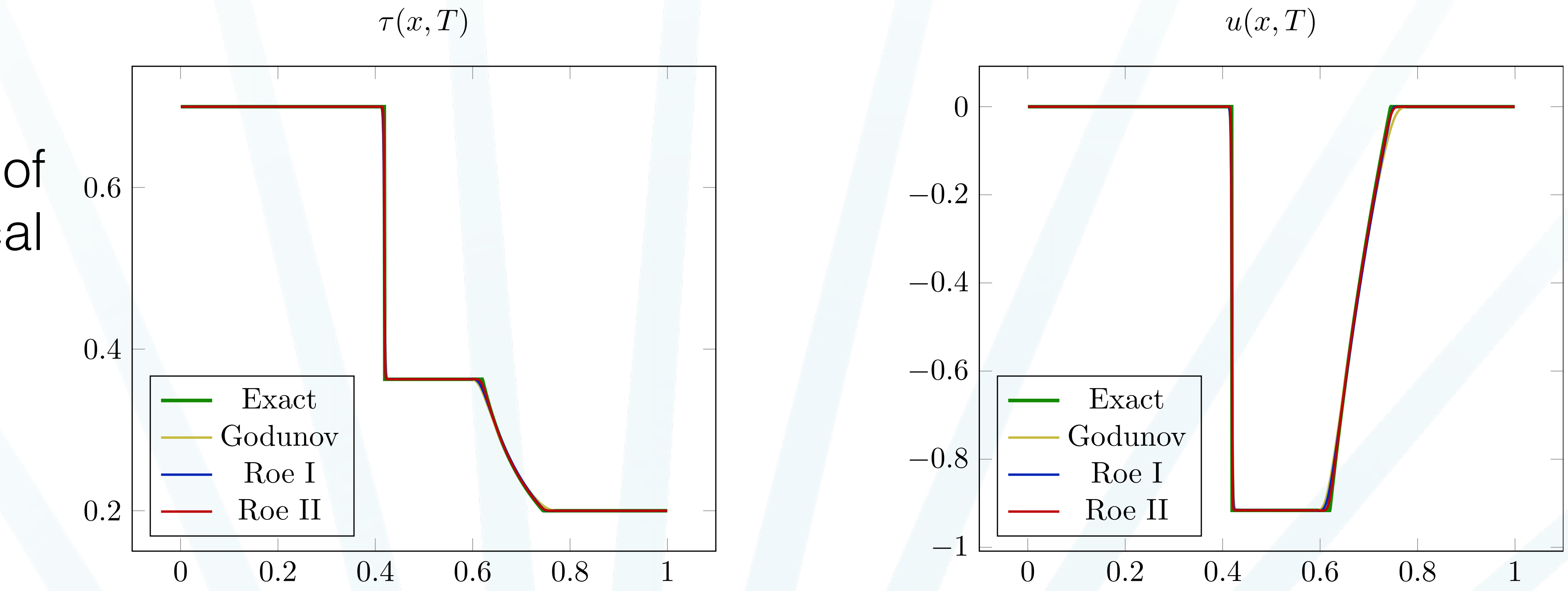

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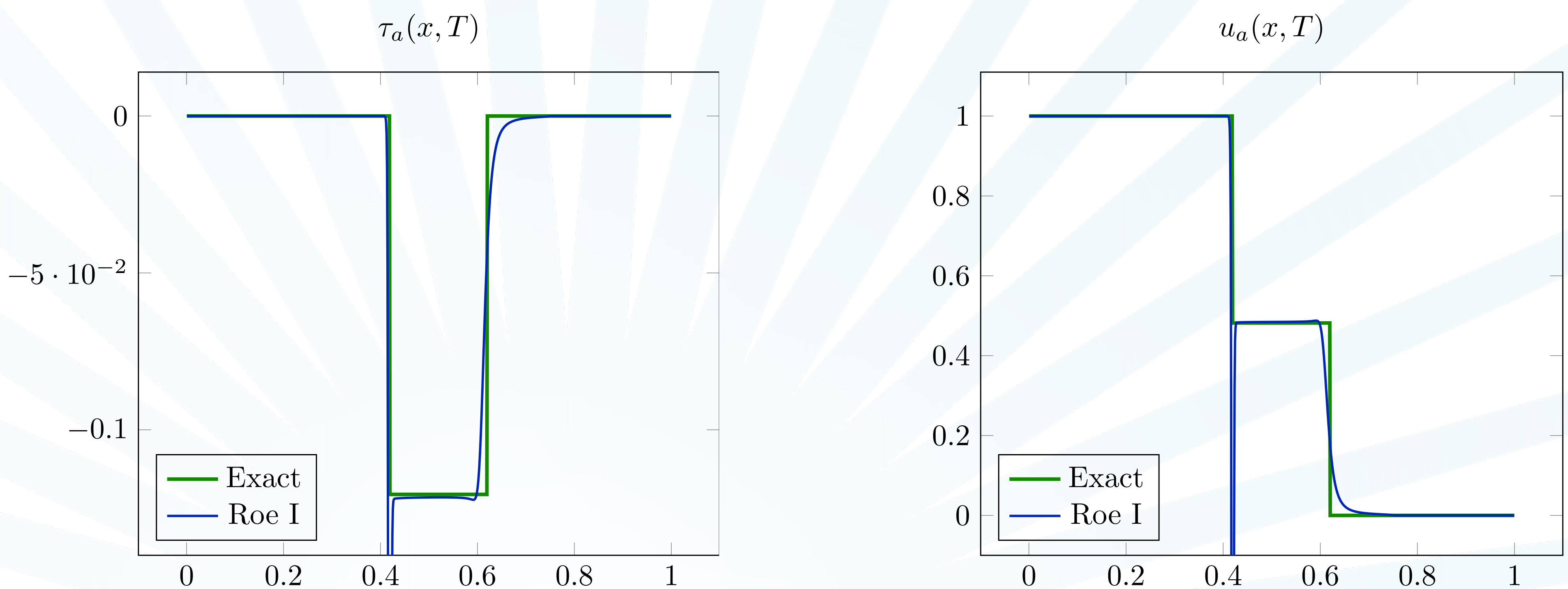
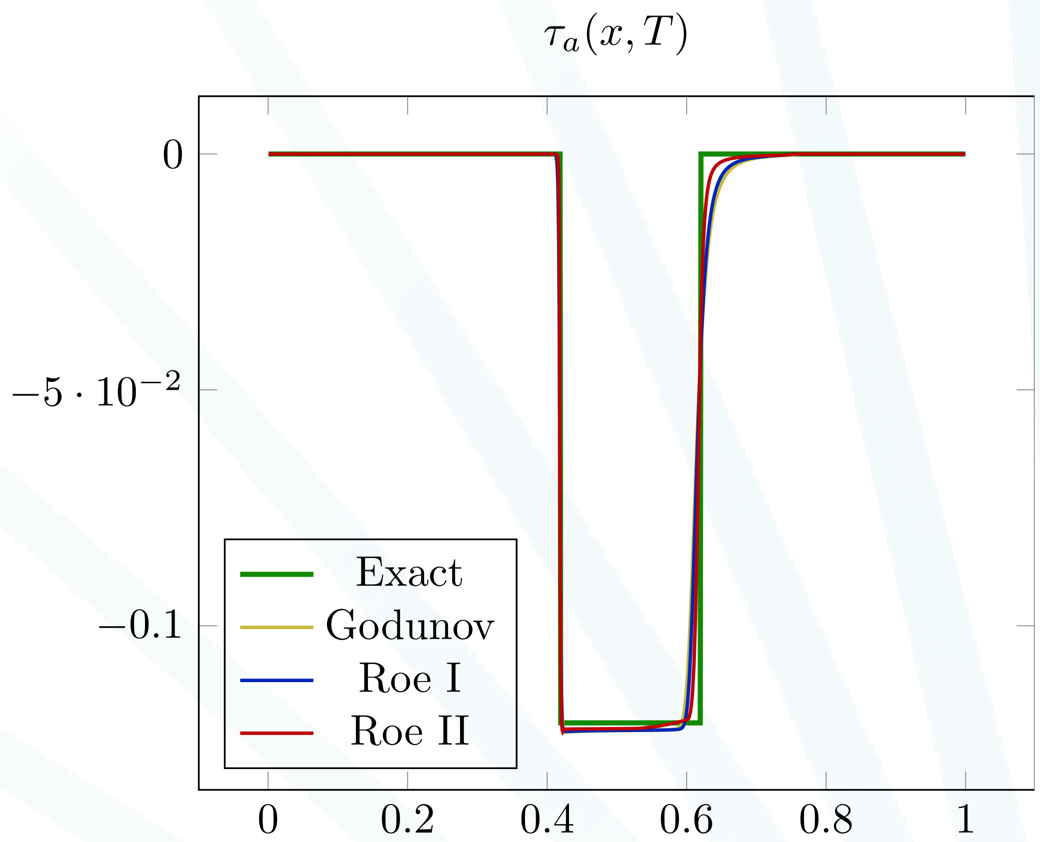
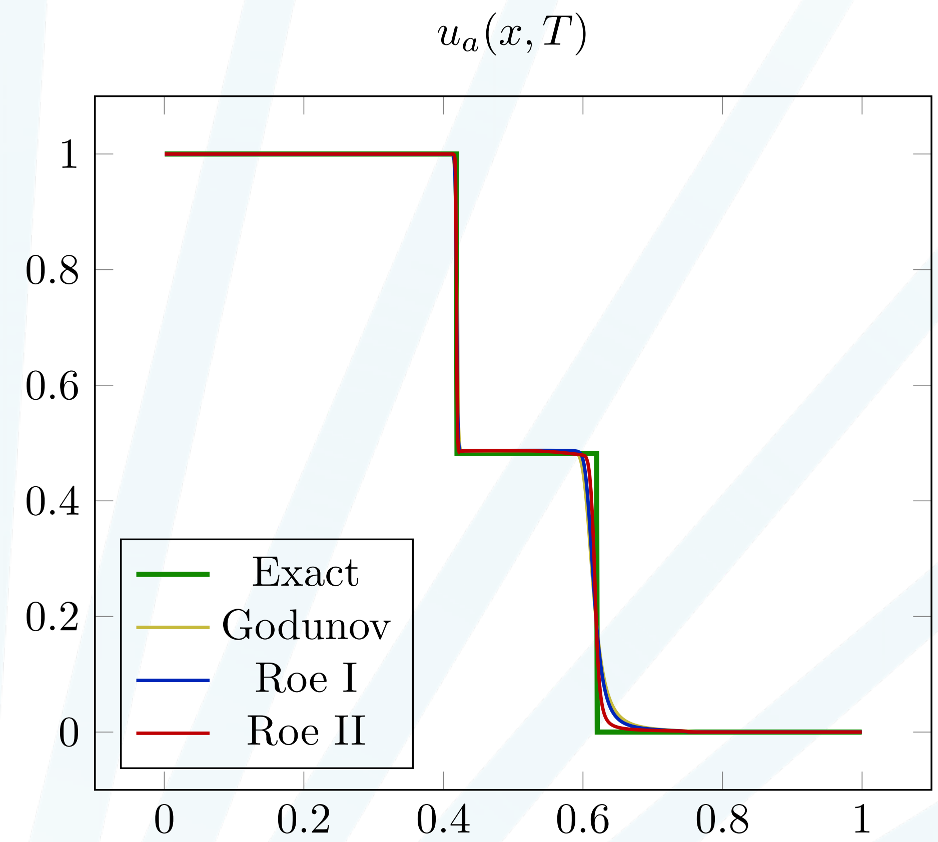


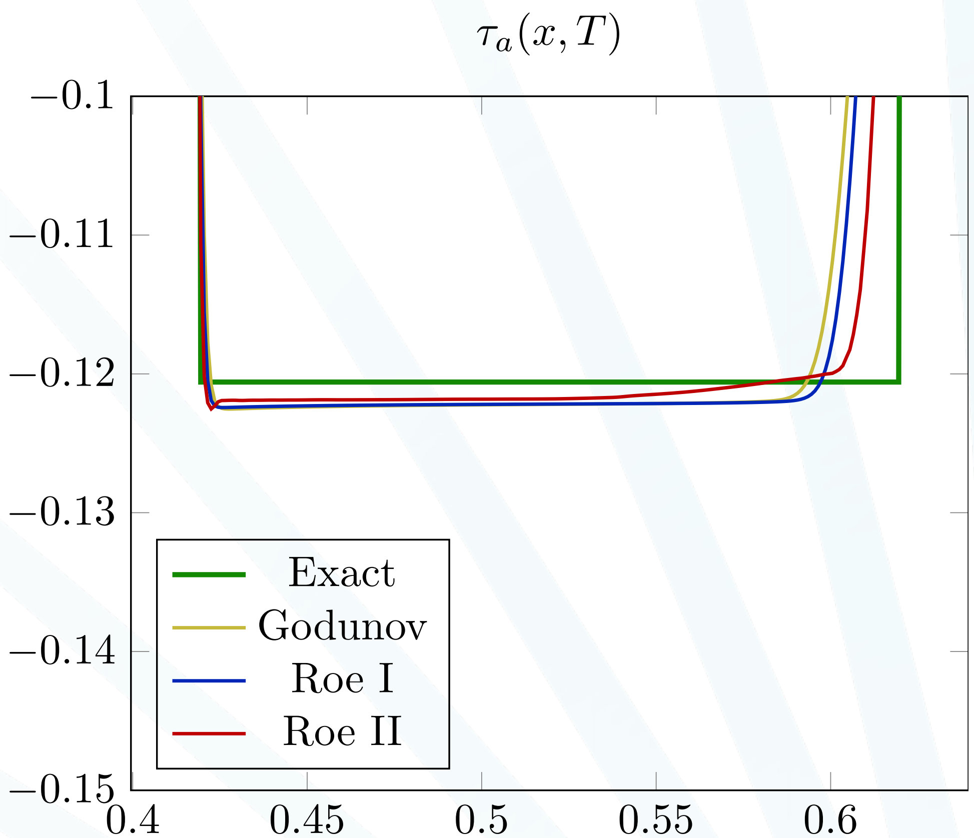
The same schemes **with source term** for the sensitivity:

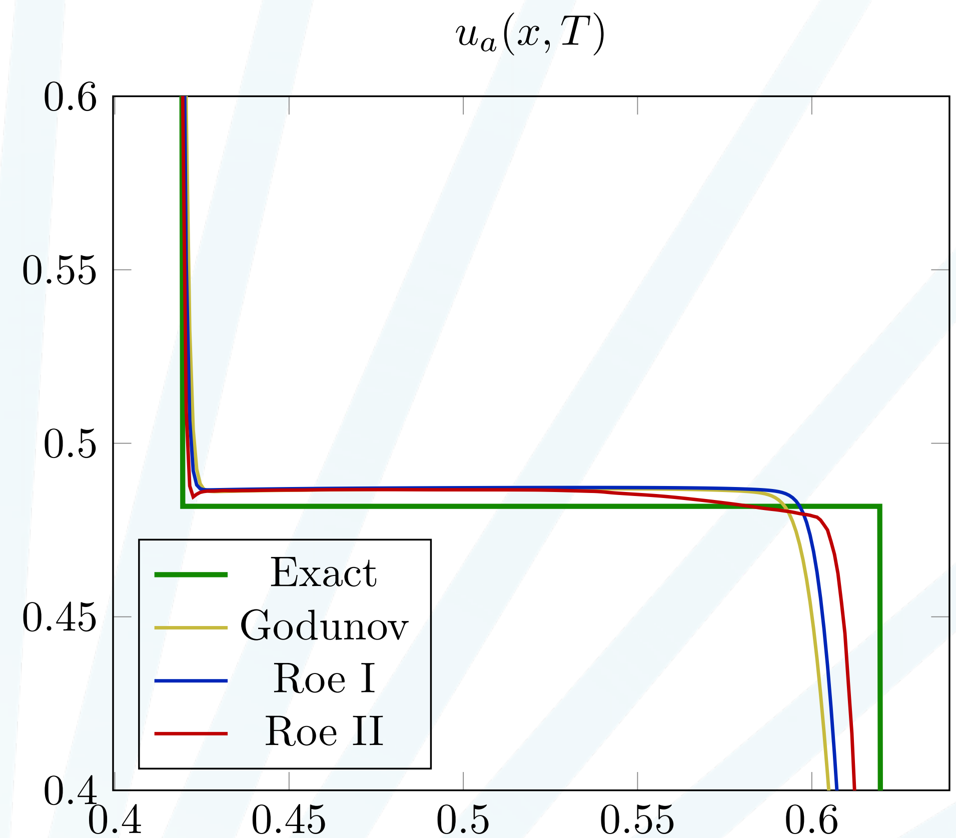




Problems:
- the rarefaction is a discontinuity for the sensitivity,
- the sensitivity value in the star zone is not correct.

The same schemes **with source term** for the sensitivity:





Problems:

- the rarefaction is a discontinuity for the sensitivity,
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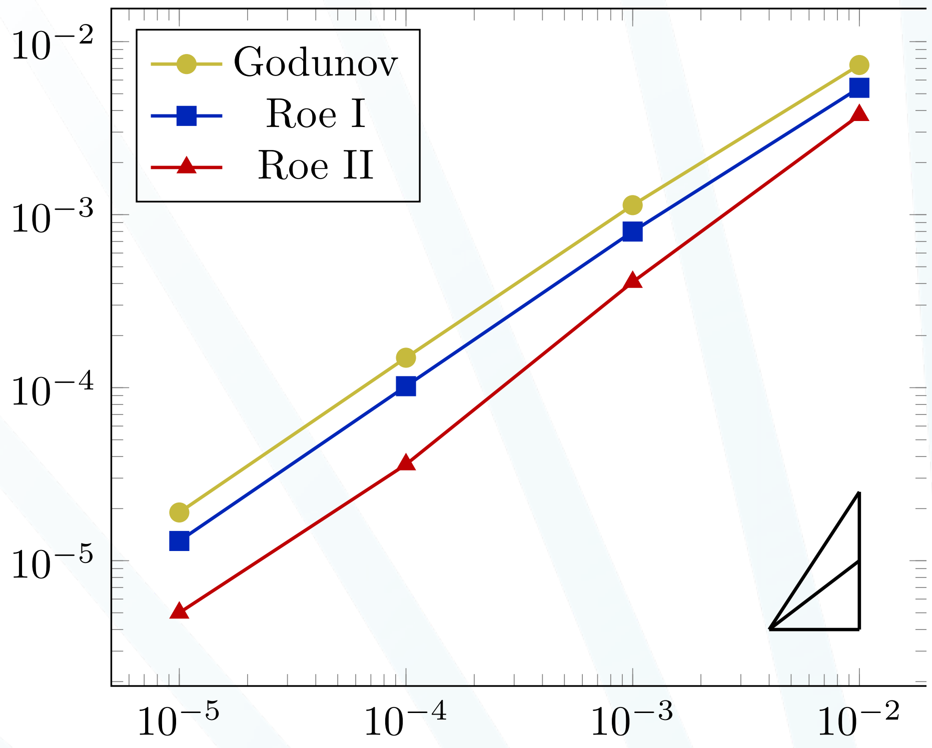
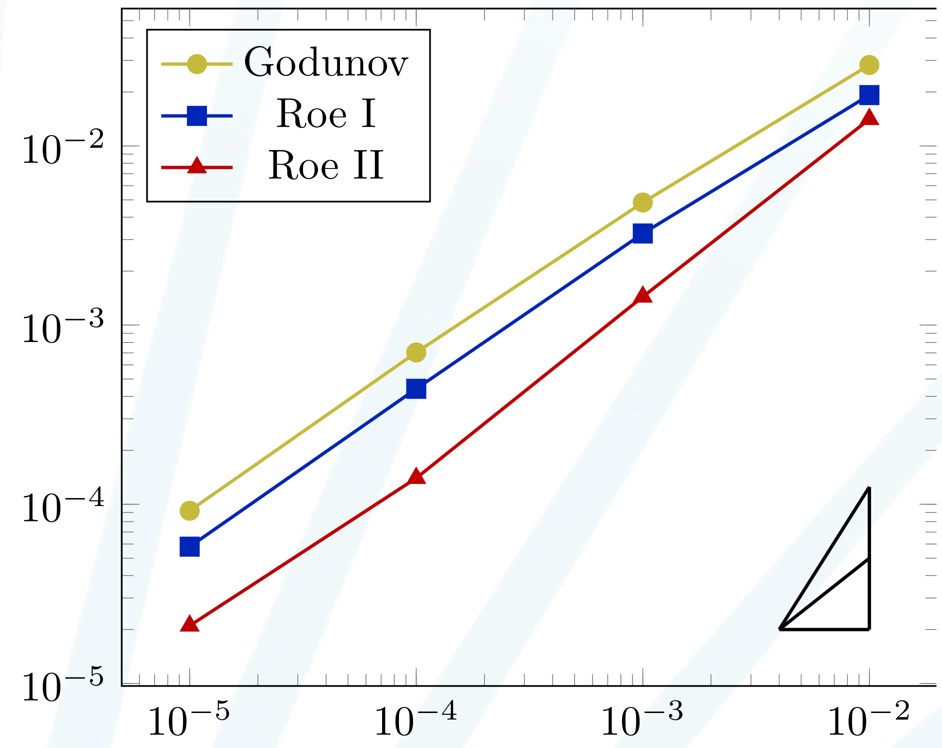
Convergence

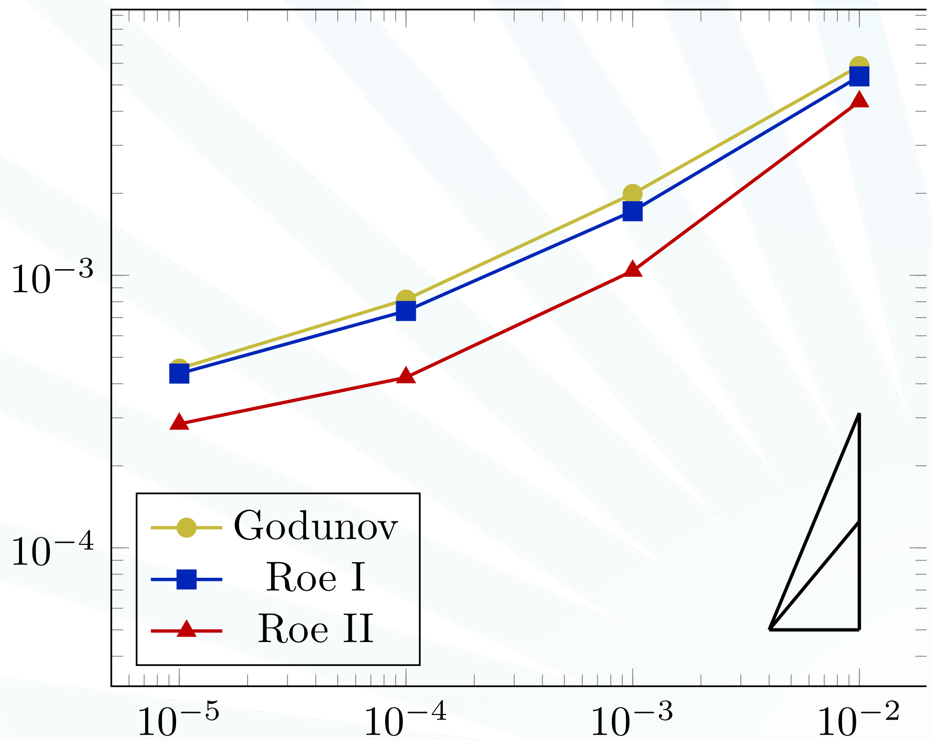
$\|\tau(x,T) - \tau_{ex}(x,T)\|_{L^1}$



$\|u(x,T) - u_{ex}(x,T)\|_{L^1}$



$\|\tau_a(x,T) - \tau_{a,ex}(x,T)\|_{L^1}$



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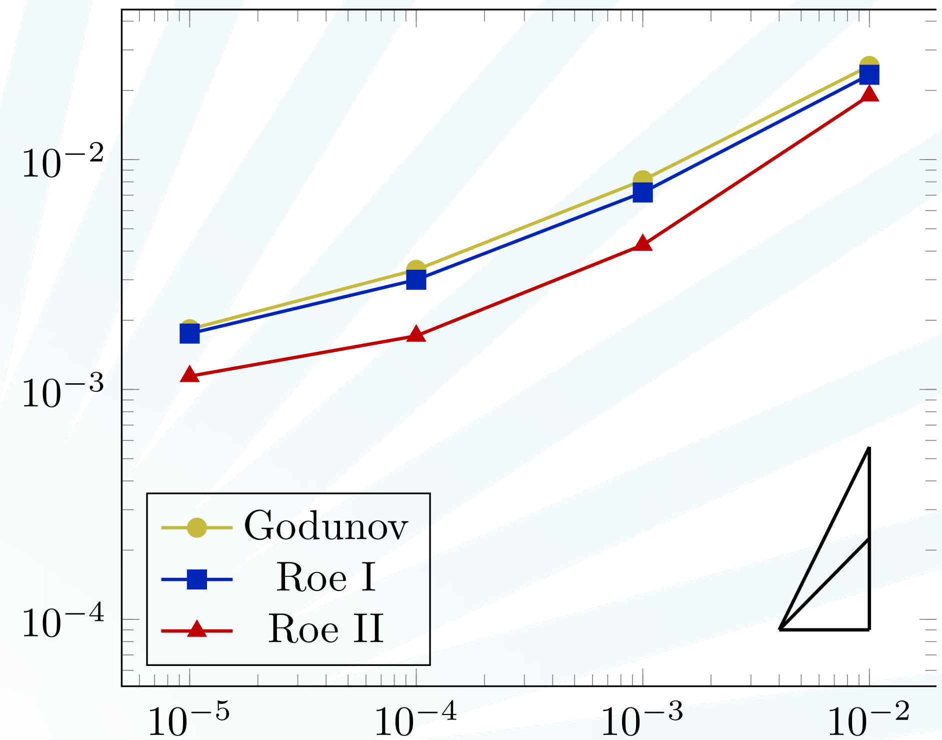
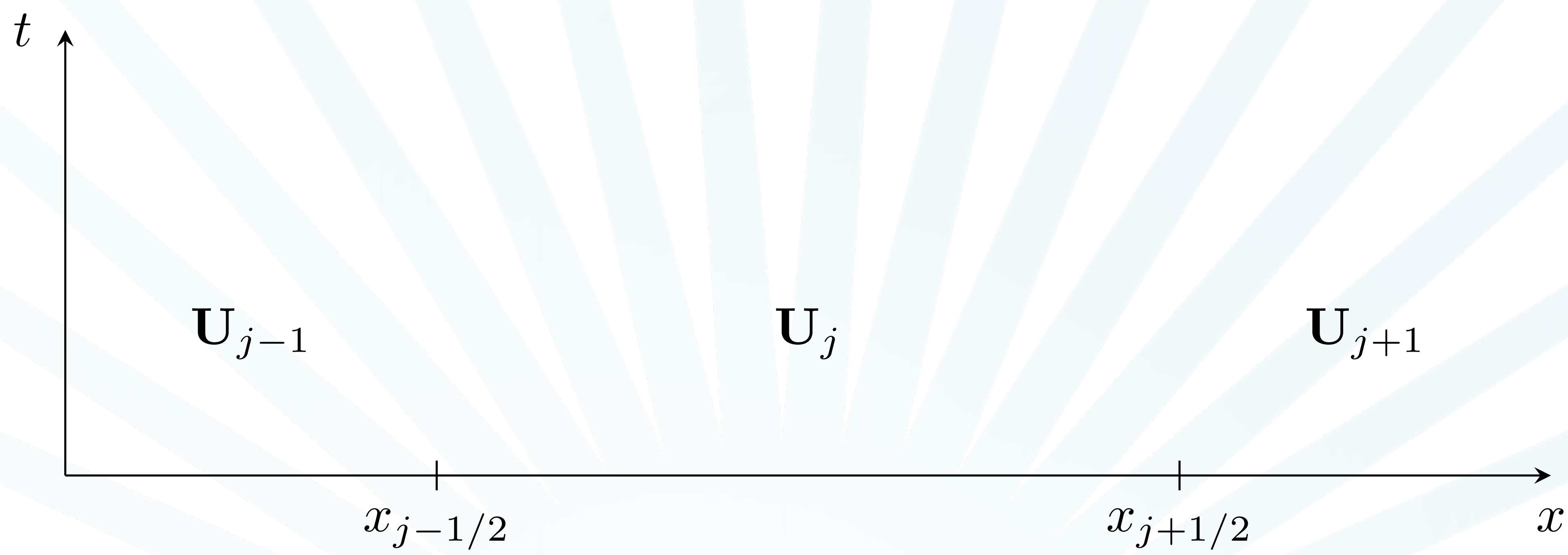


Numerical Scheme without diffusion

Scheme without numerical diffusion

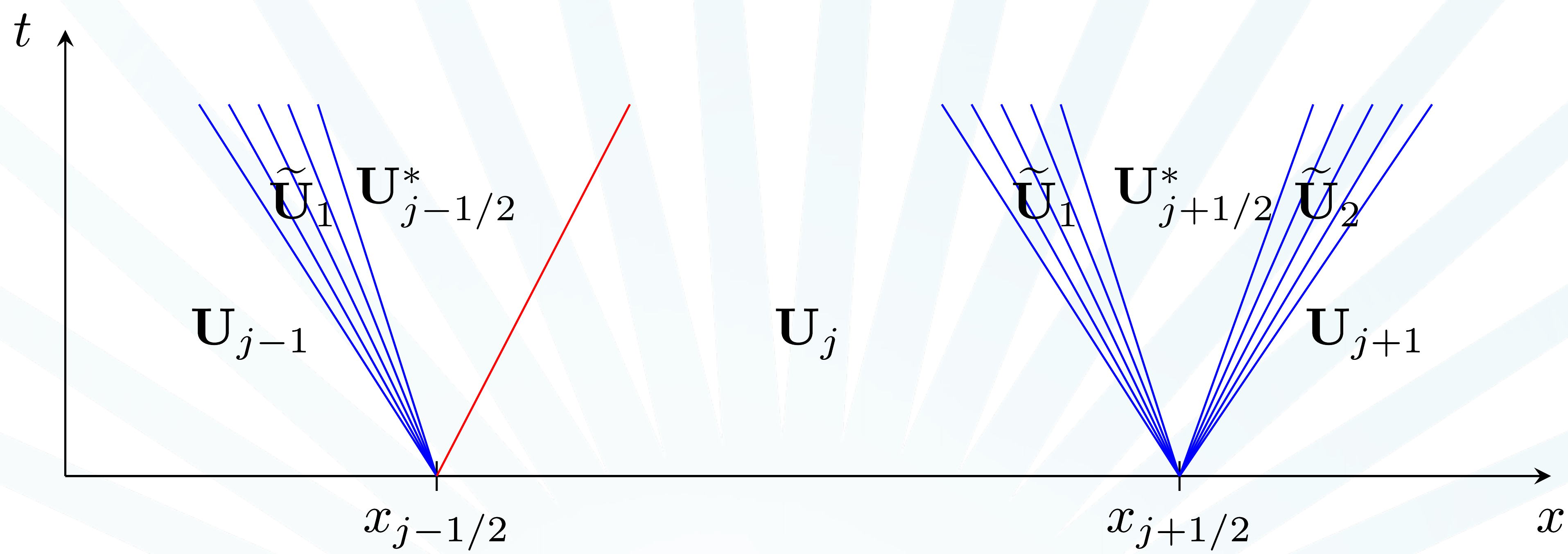
Step 0 : initial data discretisation



Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

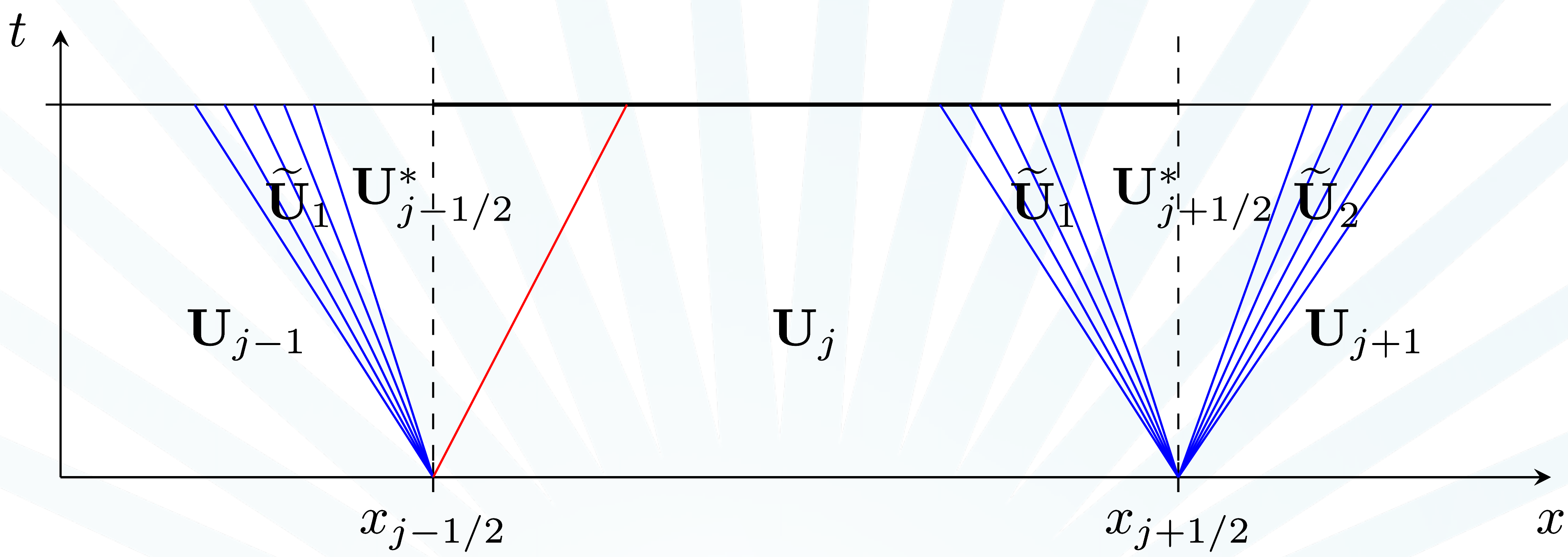


Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : average



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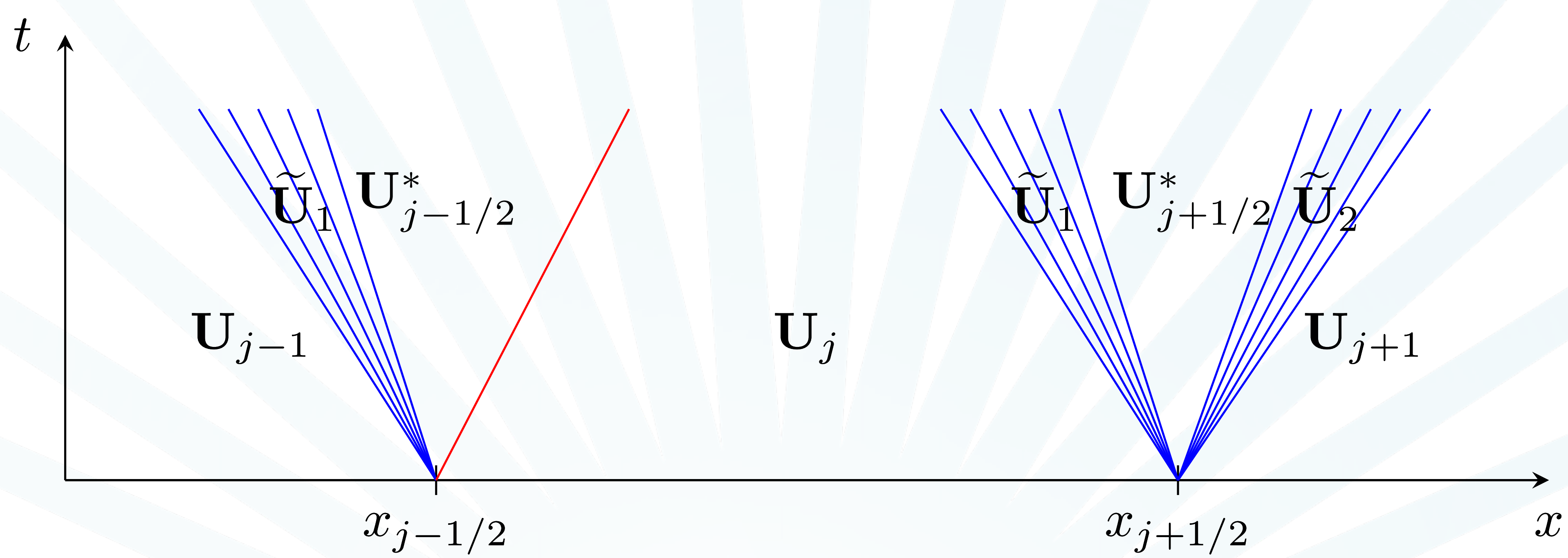
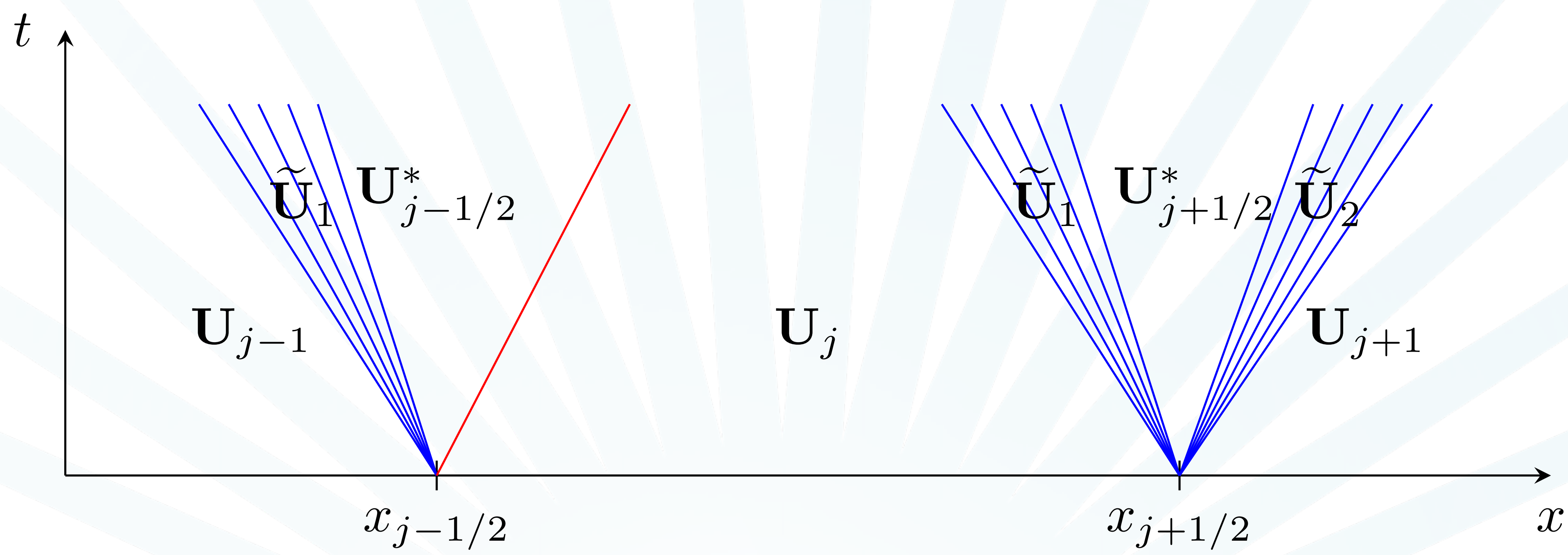


Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed



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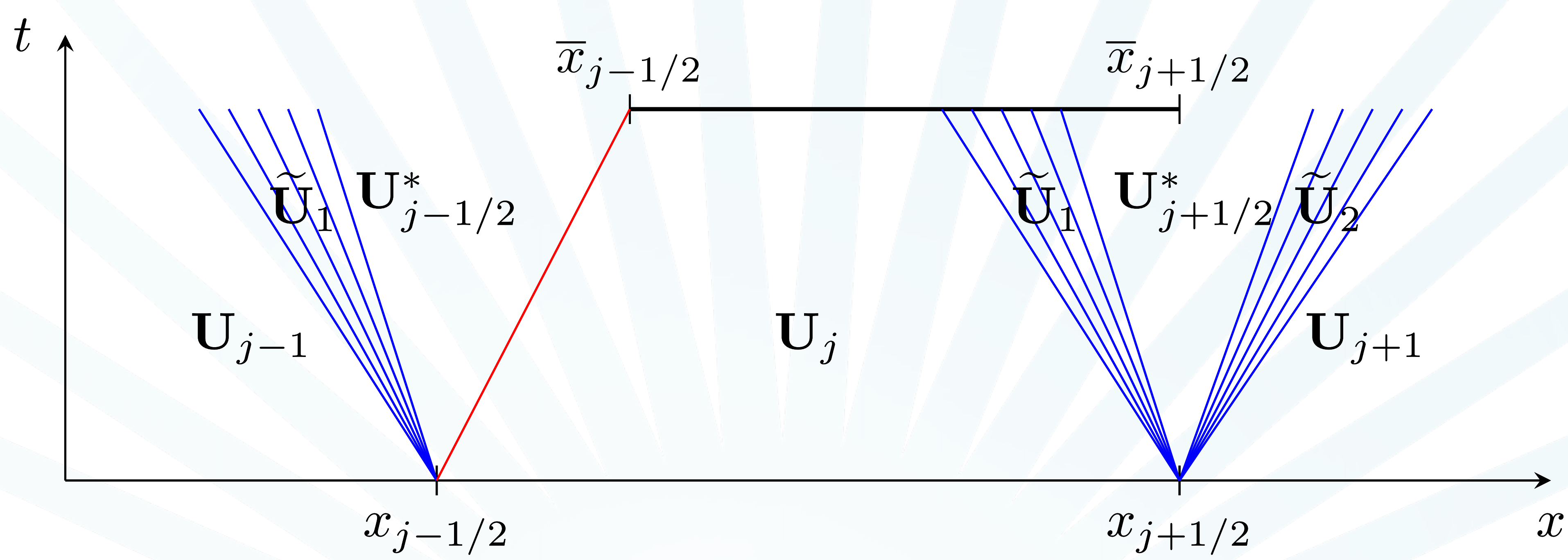

$$\overline{x}_{j-1/2} = x_{j-1/2} + \sigma_{j-1/2}\Delta t$$

Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed

Step 3 : projection on the initial mesh

Scheme without numerical diffusion

Step 0 : initial data discretisation

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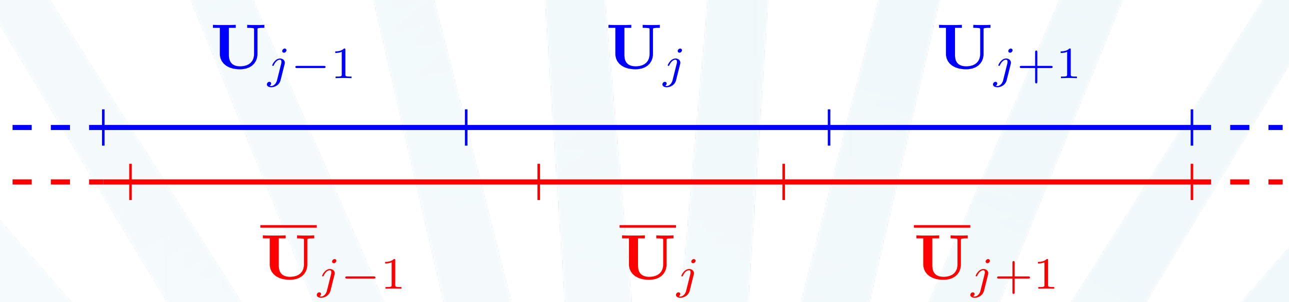
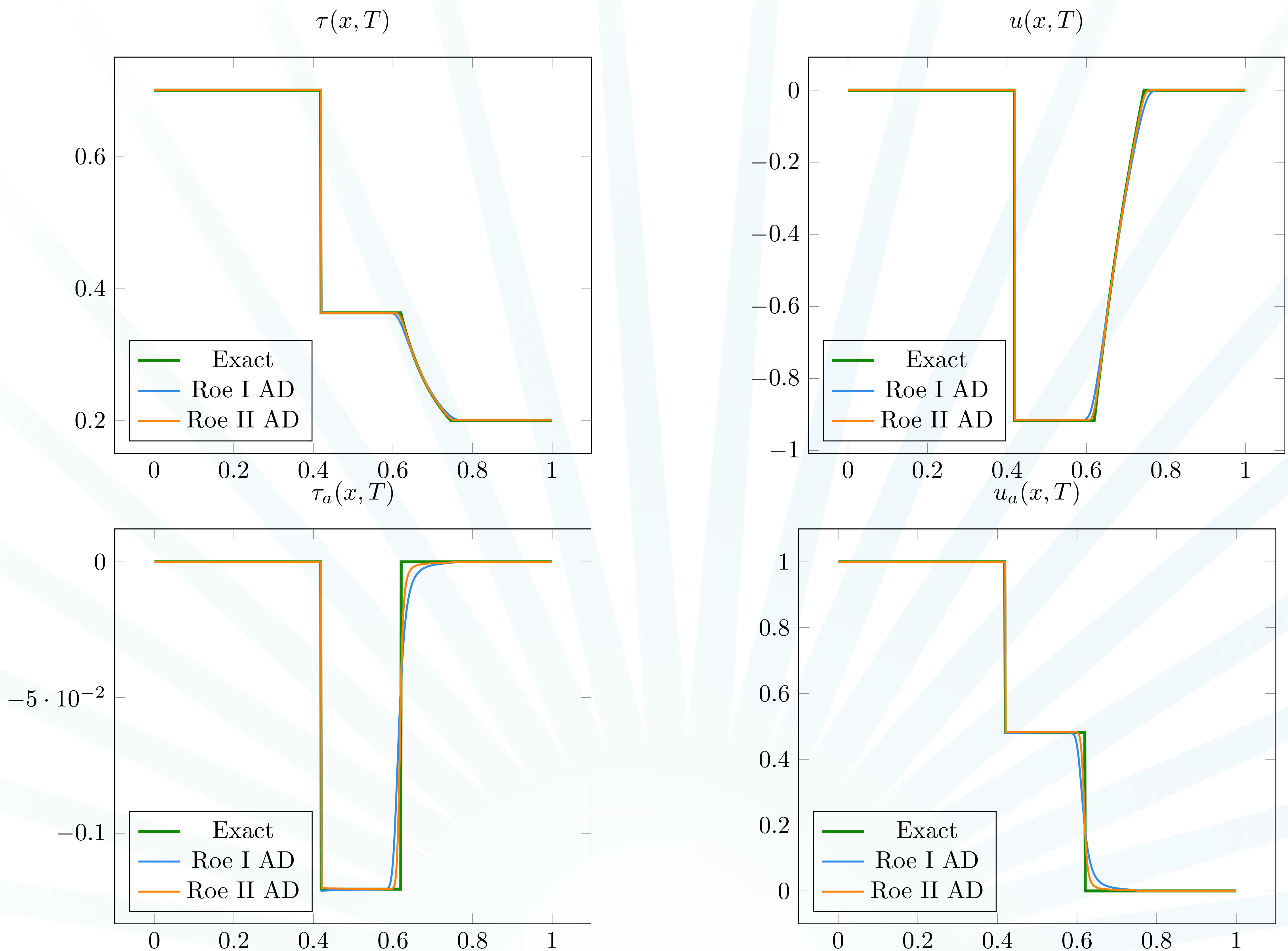


$$\mathbf{U}_j = \begin{cases} \overline{\mathbf{U}}_{j-1} & \text{if } a \in \left(0, \frac{\Delta t}{\Delta x} \max(\sigma_{j-1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j} & \text{if } a \in \left[\frac{\Delta t}{\Delta x} \max(\sigma_{j-1/2}, 0), 1 + \frac{\Delta t}{\Delta x} \min(\sigma_{j+1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j+1} & \text{if } a \in \left[1 + \frac{\Delta t}{\Delta x} \min(\sigma_{j+1/2}, 0), 1\right). \end{cases}$$

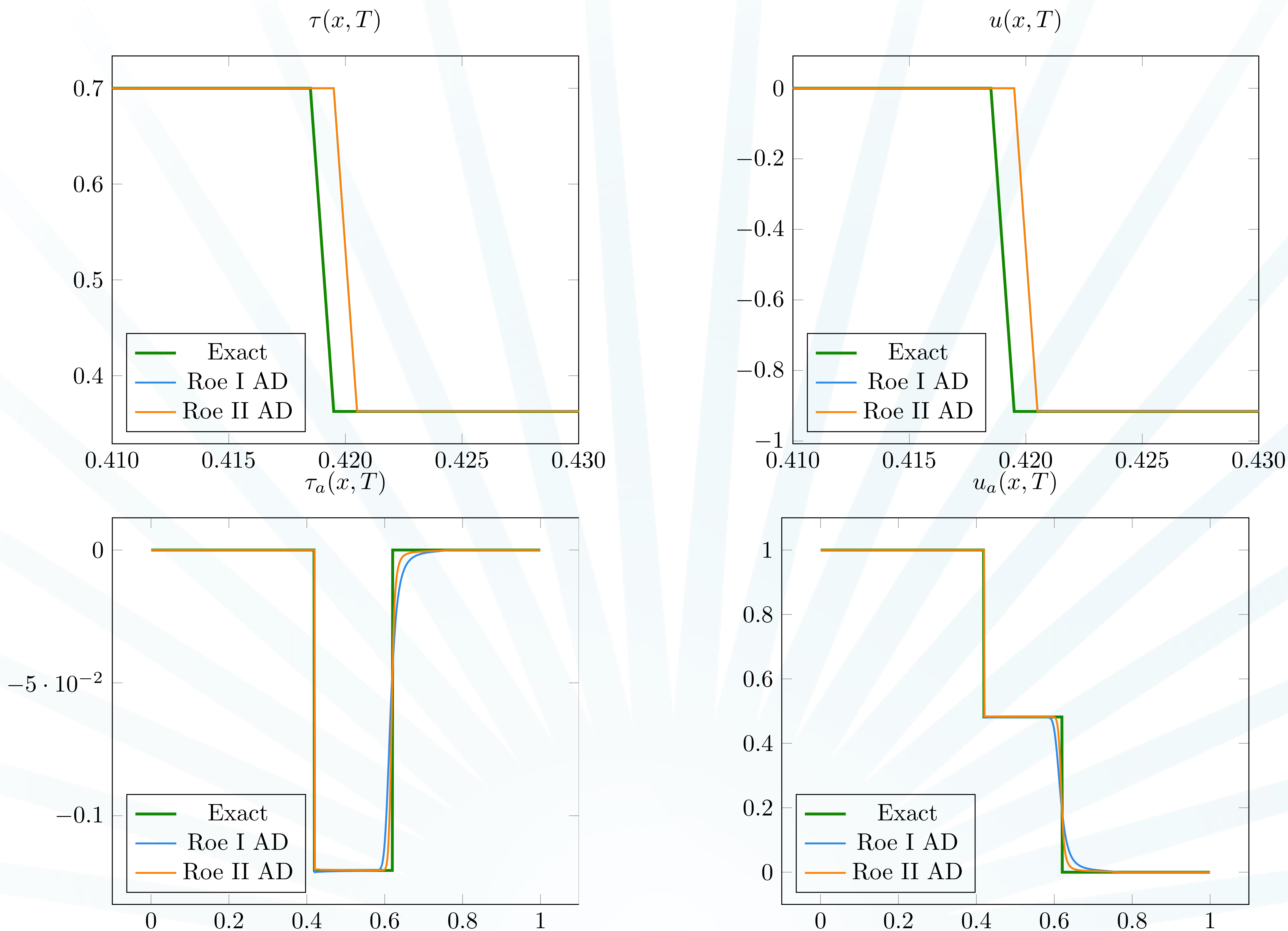
$$a \sim \mathcal{U}([0,1])$$

Results



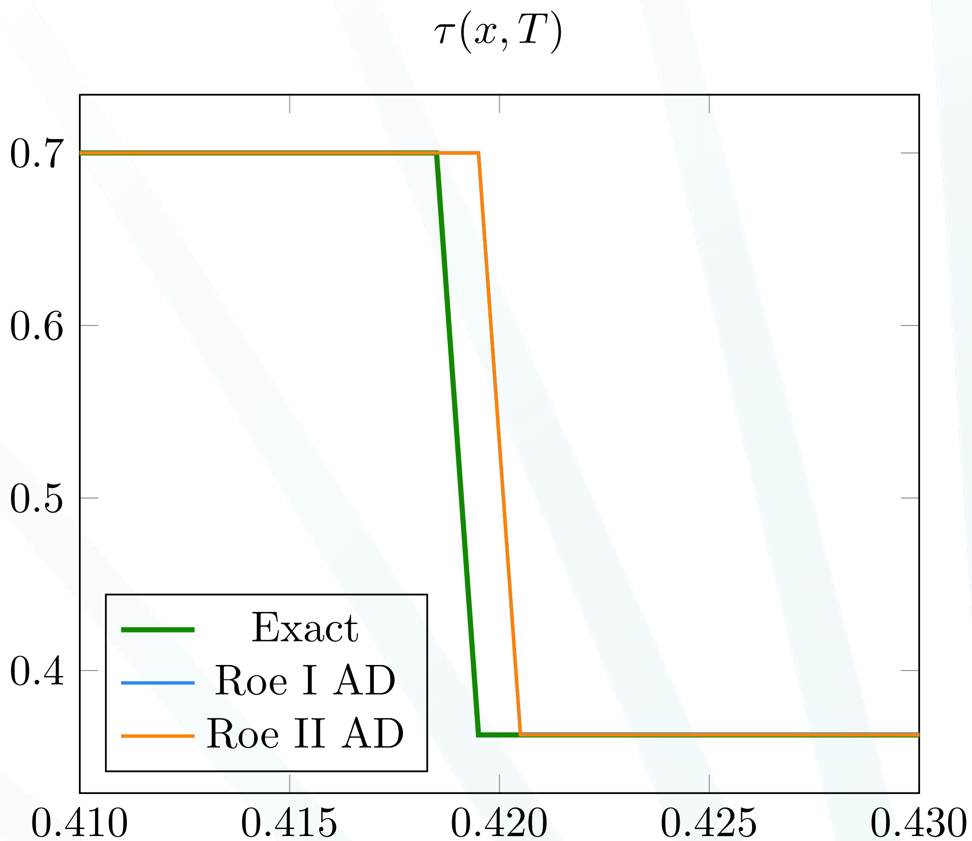
$\tau(x,T)$
$u(x,T)$
$\tau_a(x,T)$
$u_a(x,T)$
Exact
Roe I AD
Roe II AD
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
$-5 \cdot 10^{-2}$
−0.1
1
0.8
0.2

Results



τ(x, T)
u(x, T)
τa(x, T)
ua(x, T)
Exact
Roe I AD
Roe II AD
0.410
0.415
0.420
0.425
0.430
0.7
0.6
0.5
0.4
0
−0.2
−0.4
−0.6
−0.8
−1
−5 · 10−2
−0.1
1
0.8
0.6
0.4
0.2

Results


$\tau(x,T)$
0.7
0.6
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Exact
Roe I AD
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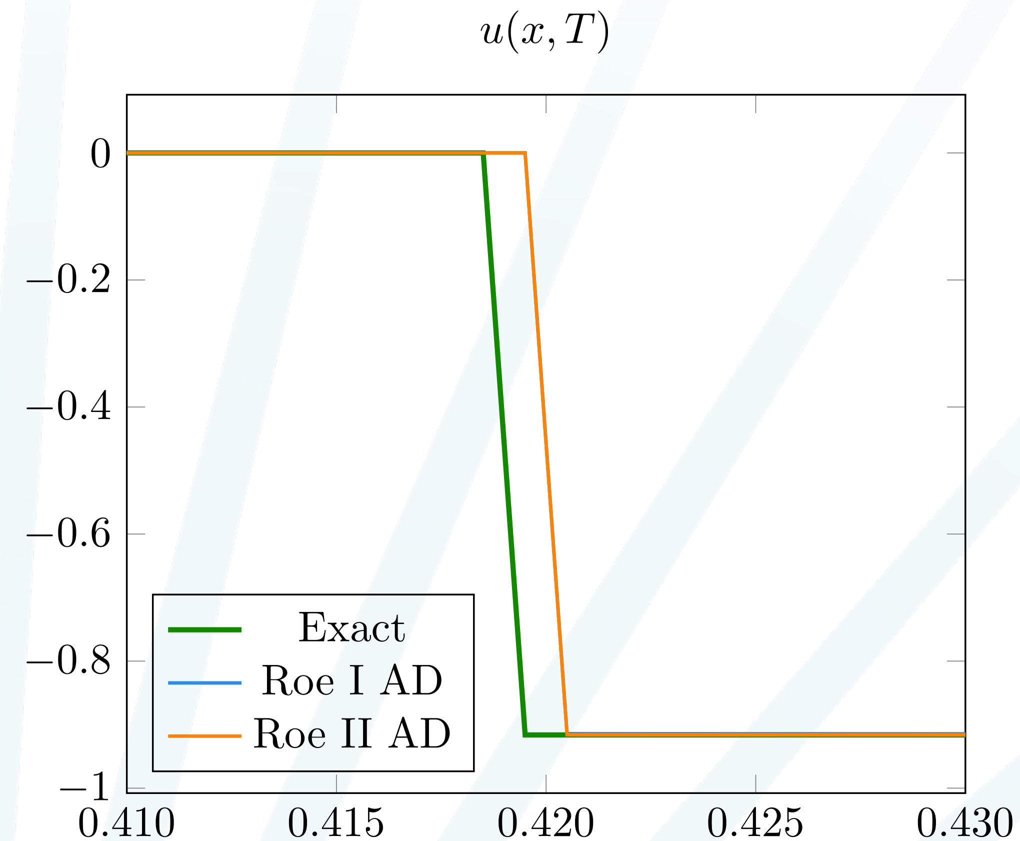
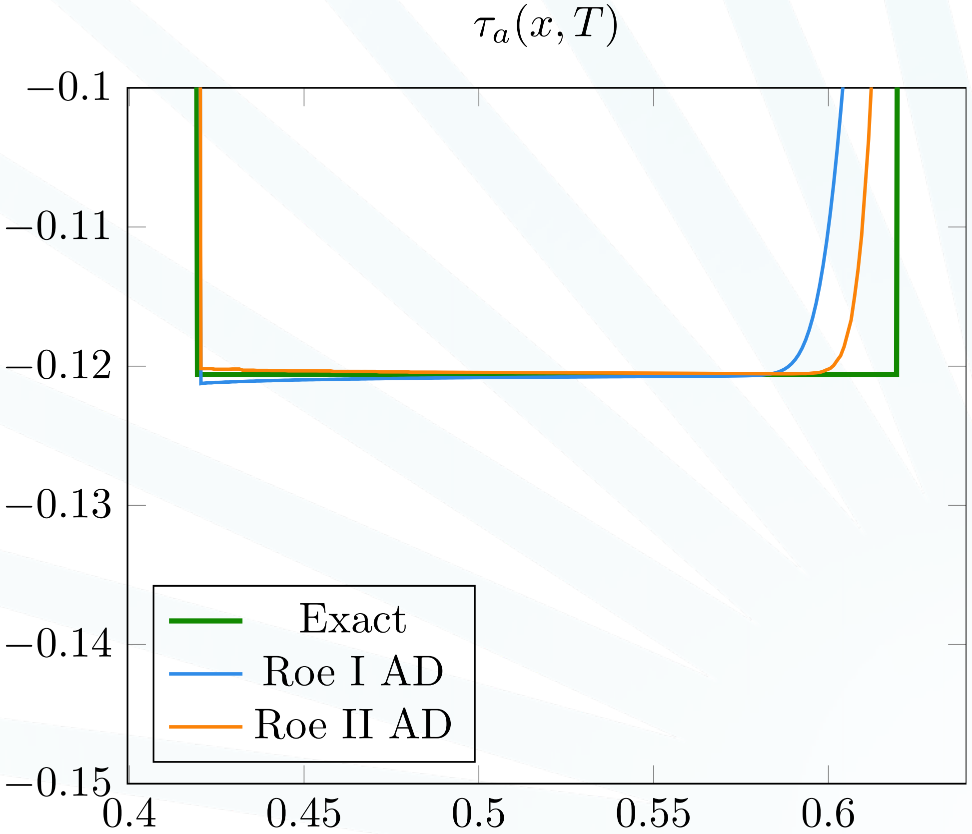
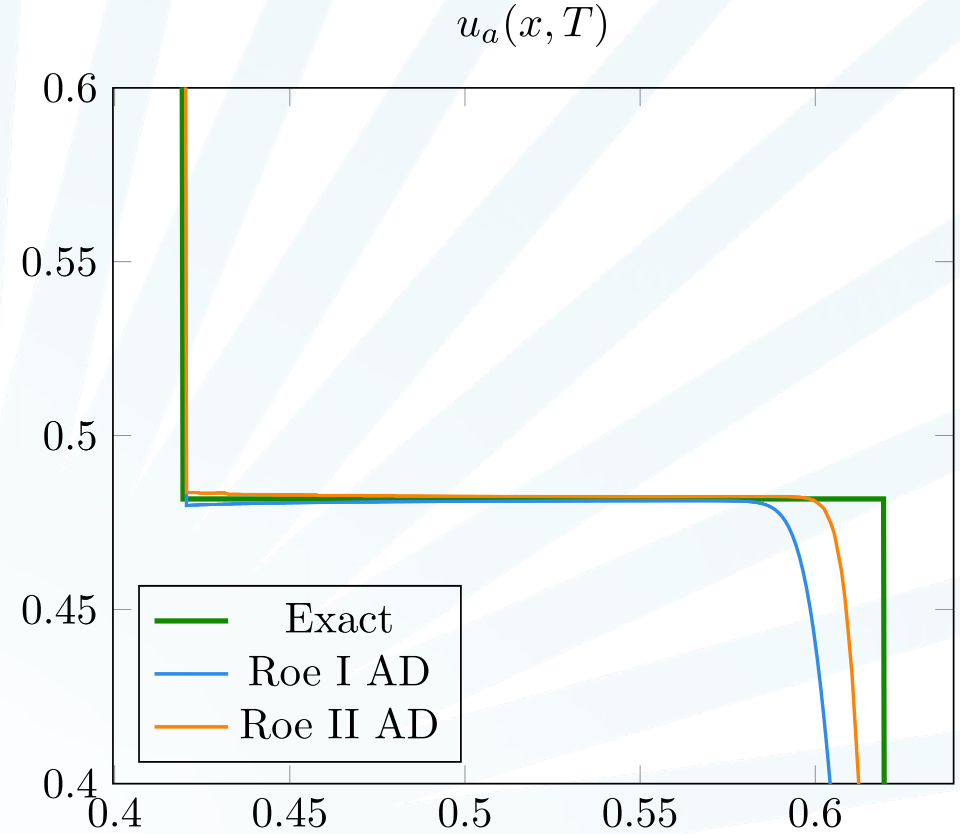
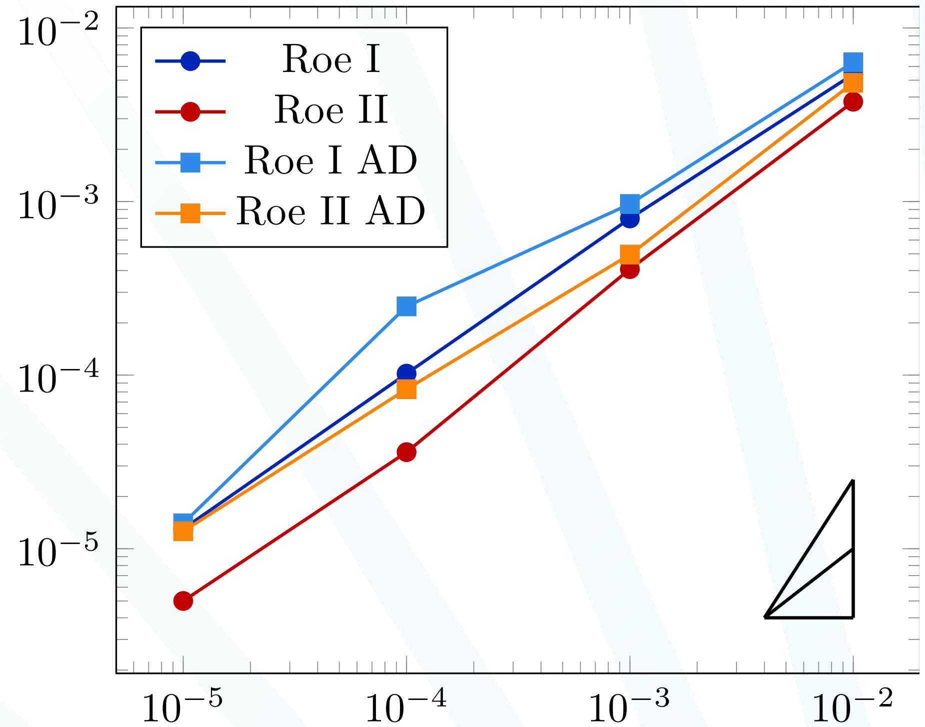

$u(x,T)$
0
−0.2
−0.4
−0.6
−0.8
−1
0.410
0.415
0.420
0.425
0.430
Exact
Roe I AD
Roe II AD

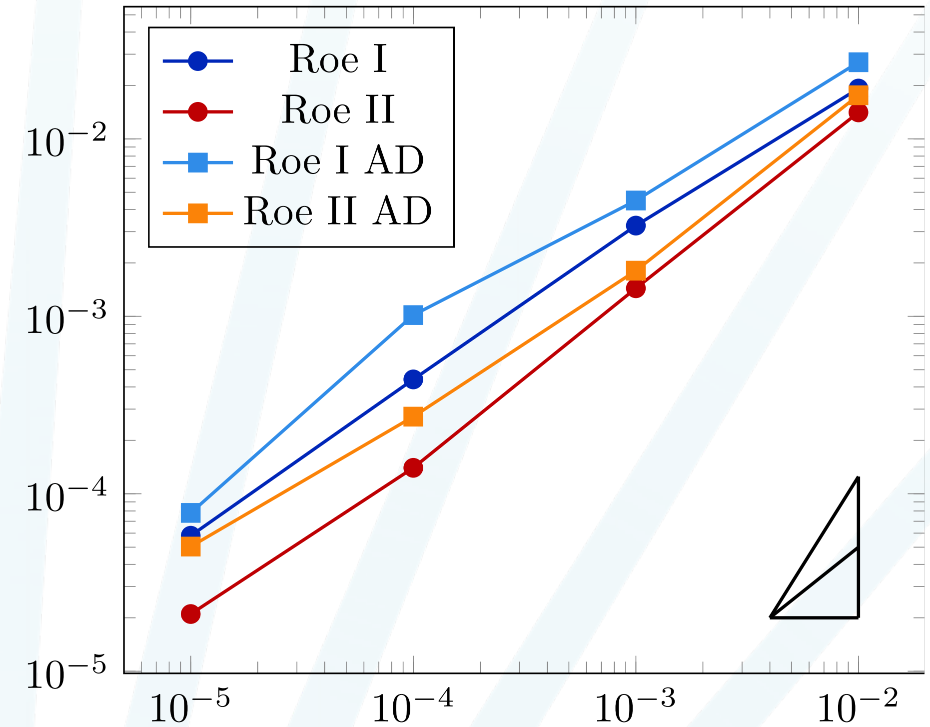

$\tau_a(x,T)$
−0.1
−0.11
−0.12
−0.13
−0.14
−0.15
0.4
0.45
0.5
0.55
0.6
Exact
Roe I AD
Roe II AD


$u_a(x,T)$
0.6
0.55
0.5
0.45
0.4
0.4
0.45
0.5
0.55
0.6
Exact
Roe I AD
Roe II AD

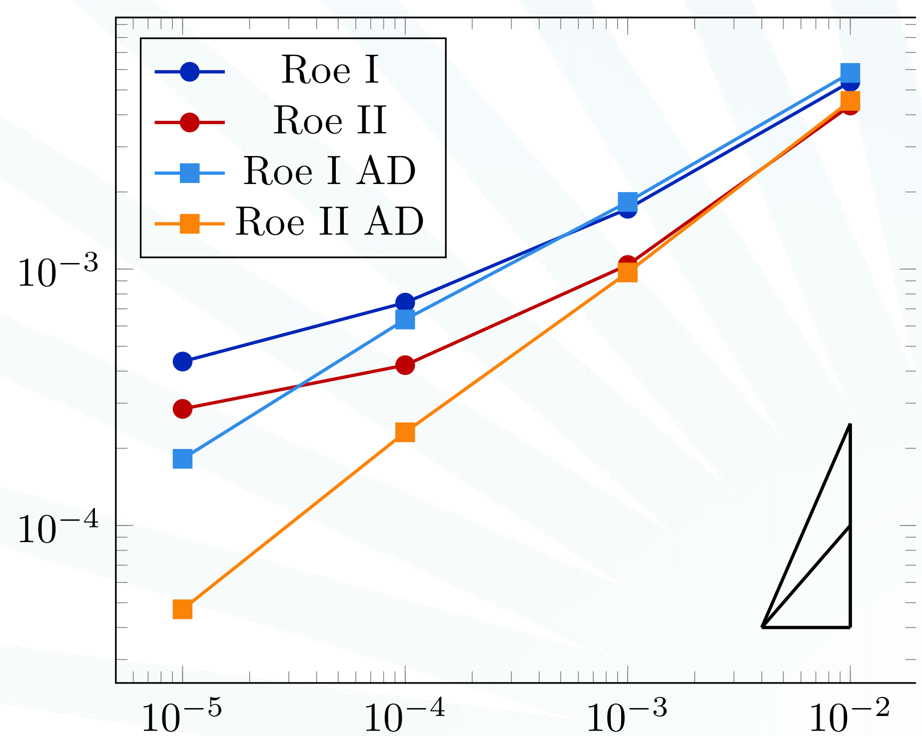
$\|\tau(x,T)-\tau_{ex}(x,T)\|_{L^1}$



$\|u(x,T)-u_{ex}(x,T)\|_{L^1}$



$\|\tau_a(x,T)-\tau_{a,ex}(x,T)\|_{L^1}$



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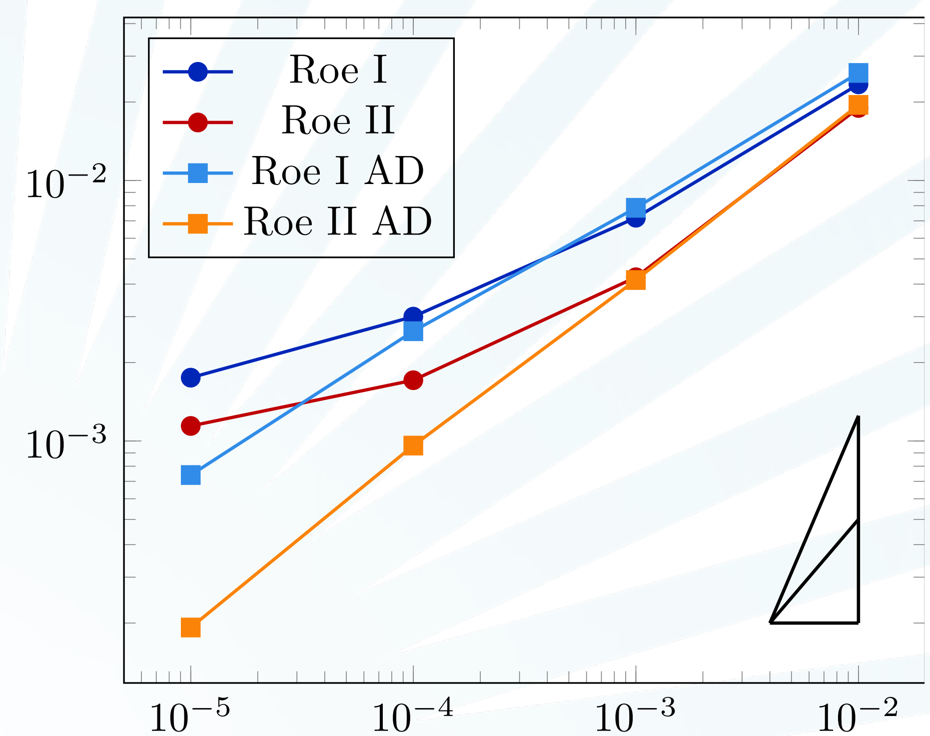
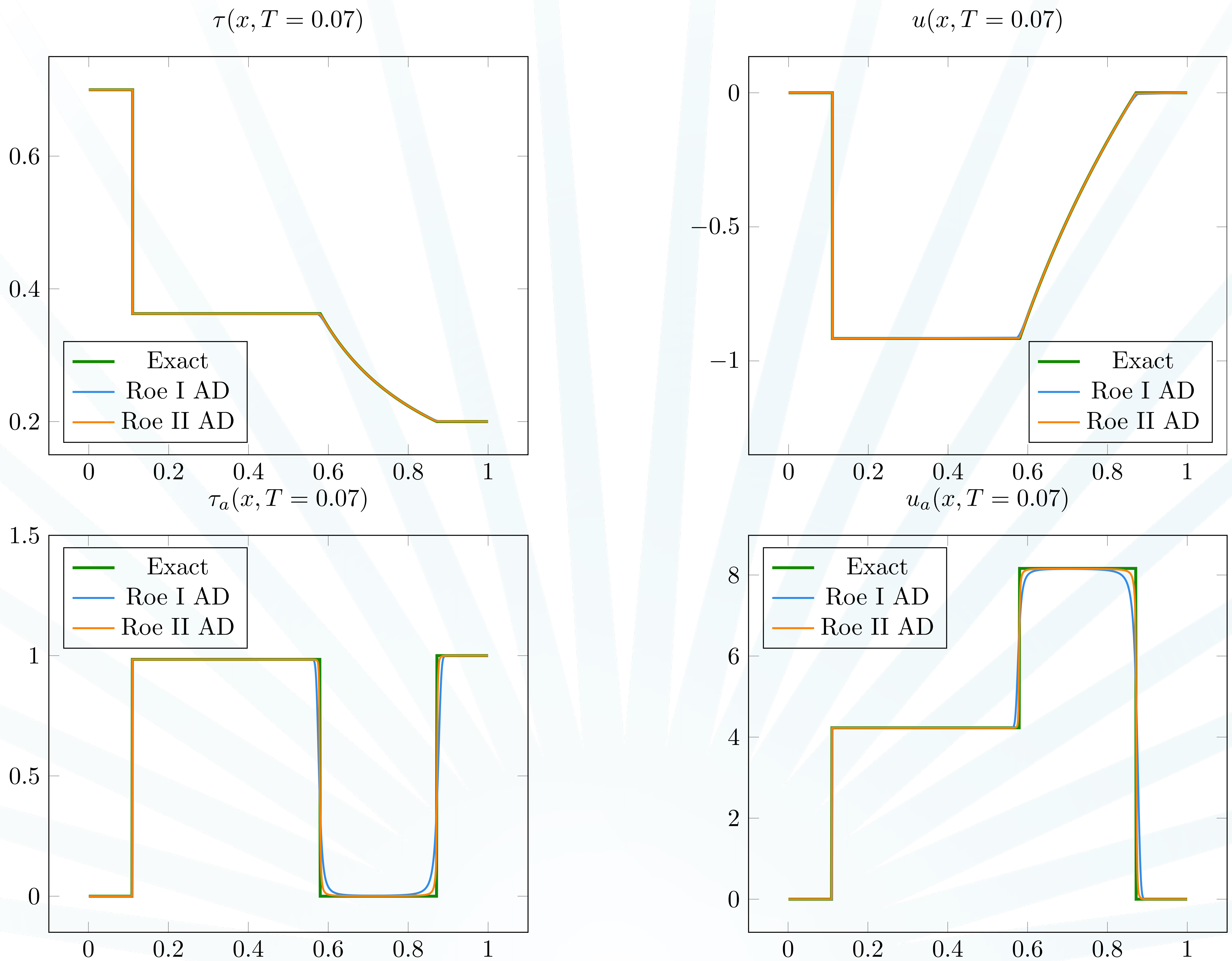


Results



Conclusion and future development

Conclusion:

- We defined a sensitivity system valid in case of discontinuous state
- The correction term is well defined
- It is necessary to control the numerical diffusion in the shock

Future development:

- Extension to the Euler system
- Extension to 2D
- Applications

Thank you

for your attention!