Sensitivity analysis for nonlinear hyperbolic systems of conservation laws

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PhD defence

Camilla Fiorini [1]

Under the supervision of

Christophe Chalons[1]

Régis Duvigneau [2]

[1] LMV, UVSQ, Versailles

[2] Université Côte d'Azur, INRIA, Sophia-Antipolis

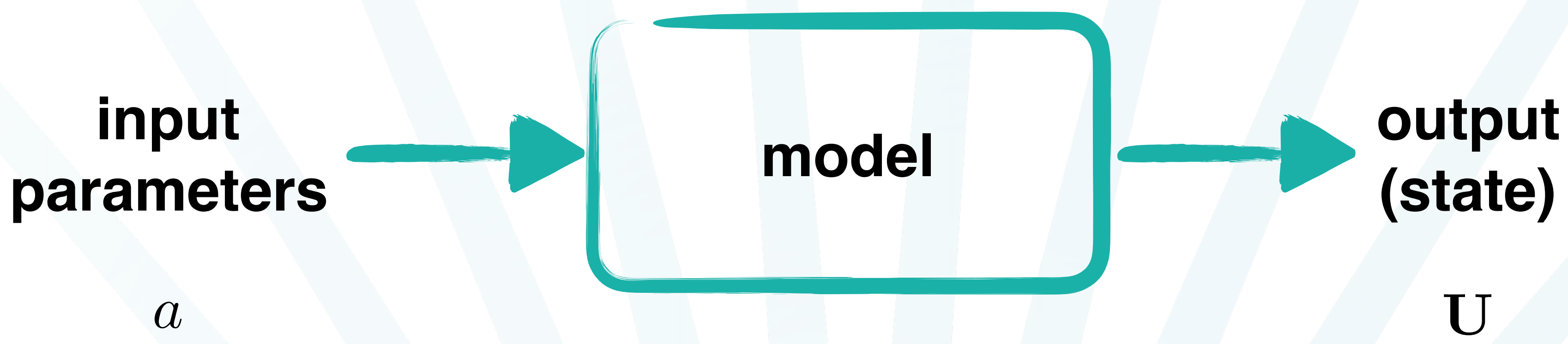
Outline of the talk

- Sensitivity analysis
- Sensitivity analysis for hyperbolic equations
- Riemann problem for the Euler equations and their sensitivity
- Classical numerical schemes
- Anti-diffusive numerical schemes
- Applications

Sensitivity Analysis

Sensitivity Analysis

Sensitivity analysis: study of how changes in the **inputs** of a model affect the **outputs**



Sensitivity: $\dfrac{\partial \mathbf{U}}{\partial a} = \mathbf{U}_a$

Applications

- Optimization [5]

Problem: $\min_{a \in \mathcal{A}} J(\mathbf{U})$, where $J(\mathbf{U}) = \frac{1}{2} b(\mathbf{U}, \mathbf{U})$ and b is bilinear.

Classical optimization techniques call for the differentiation of the cost function:

$$a^{new} = a^{old} - \alpha \frac{\partial J(\mathbf{U})}{\partial a} \qquad \frac{\partial J(\mathbf{U})}{\partial a} = b(\mathbf{U}, \mathbf{U}_a)$$

[5] Borggaard, J., & Burns, J. (1997). A PDE sensitivity equation method for optimal aerodynamic design. *Journal of Computational Physics*, *136*(2), 366-384.

[6] Duvigneau, R., & Pelletier, D. (2006). A sensitivity equation method for fast evaluation of nearby flows and uncertainty analysis for shape parameters. *International Journal of Computational Fluid Dynamics*, *20*(7), 497-512.

[7] Delenne, C. (2014). Propagation de la sensibilité dans les modèles hydrodynamiques (HDR, Montpellier II).

Applications

- Optimization [5]

- Quick evaluation of close solutions [6]

$$\mathbf{U}(a+\delta a) = \mathbf{U}(a) + \delta a \mathbf{U}_a(a) + o(\delta a^2)$$

[5] Borggaard, J., & Burns, J. (1997). A PDE sensitivity equation method for optimal aerodynamic design. *Journal of Computational Physics*, *136*(2), 366-384.
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Applications

- Optimization [5]
- Quick evaluation of close solutions [6]
- Uncertainty quantification [7]

First order estimates

$$\mu \qquad \mathbf{U}(\mu_a)$$
$$\sigma^2 \qquad \mathbf{U}_a(\mu_a)^T \mathbf{U}_a(\mu_a)\sigma_a^2$$

[5] Borggaard, J., & Burns, J. (1997). A PDE sensitivity equation method for optimal aerodynamic design. *Journal of Computational Physics*, *136*(2), 366-384.
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[7] Delenne, C. (2014). Propagation de la sensibilité dans les modèles hydrodynamiques (HDR, Montpellier II).

Two approaches

Discretise then differentiate	Differentiate then discretise
analytical state model	analytical state model
↓ discretise	↓ differentiate
discrete state model	analytical sensitivity model
↓ differentiate	↓ discretise
discrete sensitivity model	discrete sensitivity model

- analytical sensitivity model
- no discretisation of computational facilitators
- could lead to incoherent gradients

Standard techniques of sensitivity analysis

Standard techniques of sensitivity analysis call for the differentiation of the state system:

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 & \Omega \times (0, T), \\ \mathbf{U}(x, 0) = \mathbf{g}(x) & \Omega, \end{cases}$$

[8] Bardos, C., & Pironneau, O. (2002). A formalism for the differentiation of conservation laws. *Comptes Rendus Mathematique*, *335*(10), 839-845.

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For the **Burgers' equation**:

$$\mathbf{F}(\mathbf{U}) = \frac{u^2}{2} \qquad \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = u u_a$$

This can be done under **hypotheses of regularity** of the state ***U*** [8].

If these techniques are applied to hyperbolic equations, **Dirac delta functions** will appear in the sensitivity.

[8] Bardos, C., & Pironneau, O. (2002). A formalism for the differentiation of conservation laws. *Comptes Rendus Mathematique*, *335*(10), 839-845.

Standard techniques of sensitivity analysis

From different point of view: the **global system** is **weakly hyperbolic**.

The Jacobian matrix of the global system has the following form:

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$$

matrix of the state system

For the **Burgers' equation**:

Jacobian matrix: $\begin{bmatrix} u & 0 \\ u_a & u \end{bmatrix}$

therefore it has **repeated eigenvalues**.

We proved that in the general case it is **not diagonalisable**.

Sensitivity analysis for hyperbolic equations

Definition of the source term

In order to have a sensitivity system which is valid also when the state is discontinuous, we add a correction term [9]:

$$\begin{cases} \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S} & \Omega \times (0,T), \\ \mathbf{U}_a(x,0) = \mathbf{g}_a(x) & \Omega, \end{cases}$$

defined as follows:

$$\mathbf{S} = \sum_{k=1}^{N_s} \boldsymbol{\rho}_k \delta(x - x_{k,s}(t)),$$

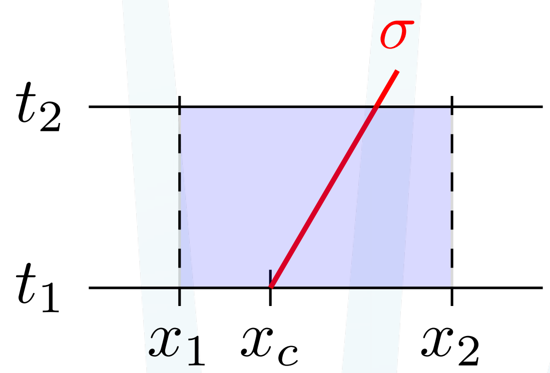
- N_s: number of discontinuities
- $x_{k,s}$: position of the k-th discontinuity
- $\boldsymbol{\rho}_k$: amplitude of the k-th correction (to be computed)

Remark: a **shock detector** is necessary to discretise such source term.

[9] Guinot, V., Delenne, C., & Cappelaere, B. (2009). An approximate Riemann solver for sensitivity equations with discontinuous solutions. *Advances in Water Resources*, *32*(1), 61-77.

Definition of the source term

To compute the amplitude of the correction, we consider an infinitesimal control volume containing a single discontinuity:



By integrating the sensitivity equations with the source term on the control volume, one has:

$$\boldsymbol{\rho}_k = (\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma_k + \mathbf{F}_a^+ - \mathbf{F}_a^-$$

Rankine-Hugoniot conditions for the state: $(\mathbf{U}^+ - \mathbf{U}^-)\sigma_k = \mathbf{F}^+ - \mathbf{F}^-$

Differentiating them w.r.t. the parameter:

$$(\mathbf{U}_a^+ - \mathbf{U}_a^-)\sigma_k + (\mathbf{U}^+ - \mathbf{U}^-)\sigma_{k,a} + \sigma_k(\partial_x \mathbf{U}^+ - \partial_x \mathbf{U}^-)\partial_a x_{k,s}(t) =$$
$$= \mathbf{F}_a^+ - \mathbf{F}_a^- + \left(\frac{\partial \mathbf{F}(\mathbf{U}^+)}{\partial \mathbf{U}}\partial_x \mathbf{U}^+ - \frac{\partial \mathbf{F}(\mathbf{U}^-)}{\partial \mathbf{U}}\partial_x \mathbf{U}^-\right)\partial_a x_{k,s}(t).$$

Finally, we obtain the following amplitude:

$$\boldsymbol{\rho}_k = (\mathbf{U}^+ - \mathbf{U}^-)\sigma_{k,a} + \sigma_k(\partial_x \mathbf{U}^+ - \partial_x \mathbf{U}^-)\partial_a x_{k,s}(t) - \left(\frac{\partial \mathbf{F}(\mathbf{U}^+)}{\partial \mathbf{U}}\partial_x \mathbf{U}^+ - \frac{\partial \mathbf{F}(\mathbf{U}^-)}{\partial \mathbf{U}}\partial_x \mathbf{U}^-\right)\partial_a x_{k,s}(t).$$

Thesis synopsis

- Chapter 1: **introduction**
- Chapter 2: **scalar case**
- Chapter 3: **the *p*-system** [1,2]
- Chapter 4: **the Euler system** [3]
- Chapter 5: **quasi 1D Euler system**
- Chapter 6: **conclusion and perspectives**
- Appendix A: **modelling of running strategies** [4]

[1] Chalons, C., Duvigneau, R., & Fiorini, C. (2017). Sensitivity analysis for the Euler equations in Lagrangian coordinates. In *International Conference on Finite Volumes for Complex Applications* (pp. 71-79). Springer, Cham.
[2] Chalons, C., Duvigneau, R. & Fiorini, C. (2017). Sensitivity analysis and numerical diffusion effects for hyperbolic PDE systems with discontinuous solutions. The case of barotropic Euler equations in Lagrangian coordinates. Submitted to *SJSC.*
[3] Fiorini, C., Chalons, C., Duvigneau, R (2018). Sensitivity equation method for Euler equations in presence of shocks applied to uncertainty quantification. Submitted to *JCP.*
[4] Fiorini, C. (2017). Optimization of Running Strategies According to the Physiological Parameters for a Two-Runner Model. *Bulletin of mathematical biology*, *79*(1), 143-162.

Riemann problem
for Euler equations
and their sensitivity

The Riemann problem for Euler equations

The Euler equations write:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x(u(\rho E + p)) = 0, \end{cases}$$

Eigenvalues:	Eigenvectors:
$\lambda_1(\mathbf{U}) = u - c,$	$\mathbf{r}_1(\mathbf{U}) = (1, u - c, H - uc)^t,$
$\lambda_2(\mathbf{U}) = u,$	$\mathbf{r}_2(\mathbf{U}) = (1, u, \frac{u^2}{2})^t,$
$\lambda_3(\mathbf{U}) = u + c.$	$\mathbf{r}_3(\mathbf{U}) = (1, u + c, H + uc)^t.$

Genuinely nonlinear

The Riemann problem for Euler equations

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Linearly degenerate

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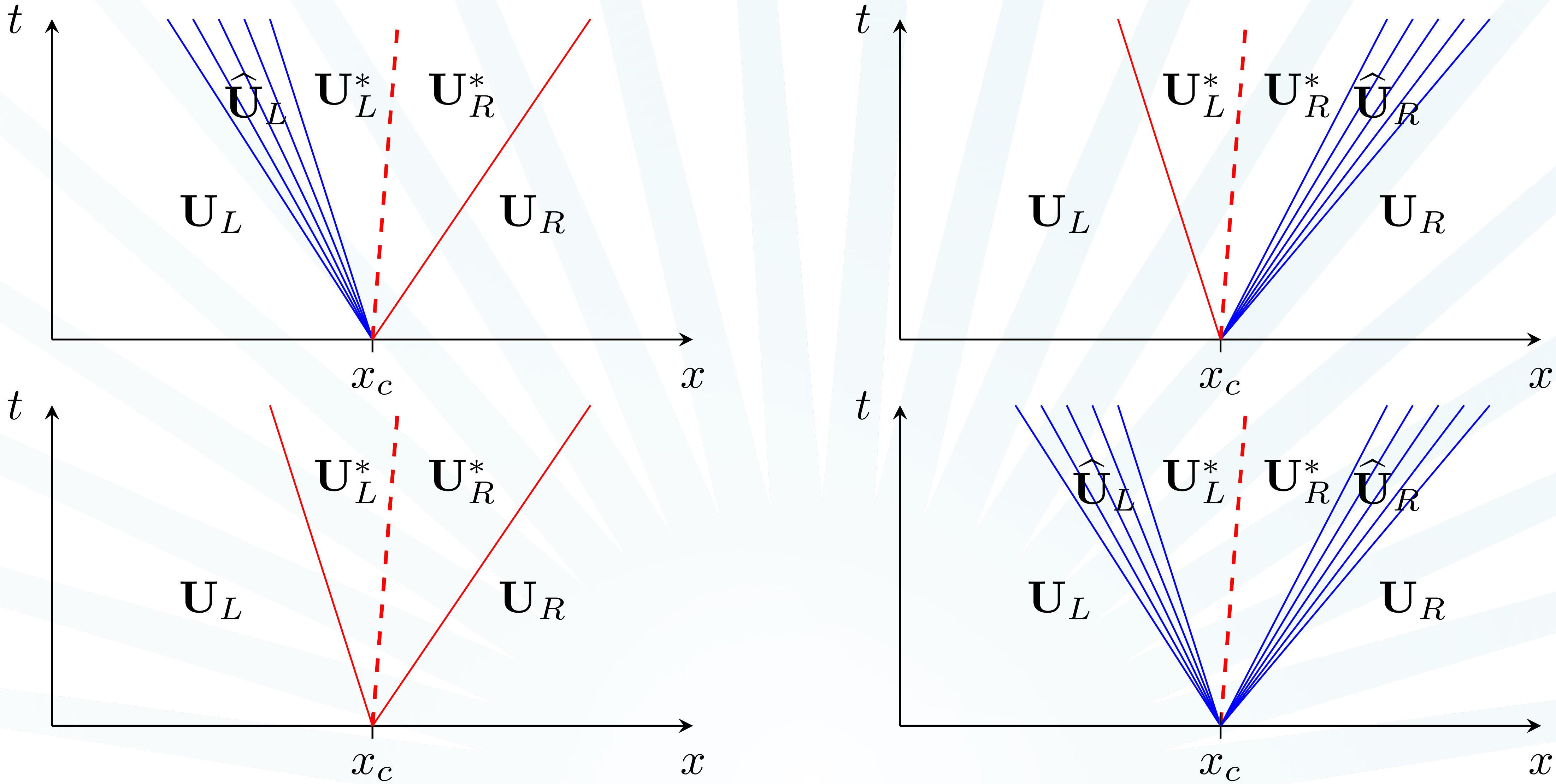

The Riemann problem for the sensitivity equations

The sensitivity system writes:

$$\begin{cases} \partial_t \rho_a + \partial_x (\rho u)_a = S_1, \\ \partial_t (\rho u)_a + \partial_x (\rho_a u^2 + 2\rho u u_a + p_a) = S_2, \\ \partial_t (\rho E)_a + \partial_x (u_a(\rho E + p) + u((\rho E)_a + p_a)) = S_3, \end{cases}$$

Eigenvalues:

$$\lambda_1(\mathbf{U}) = u - c,$$
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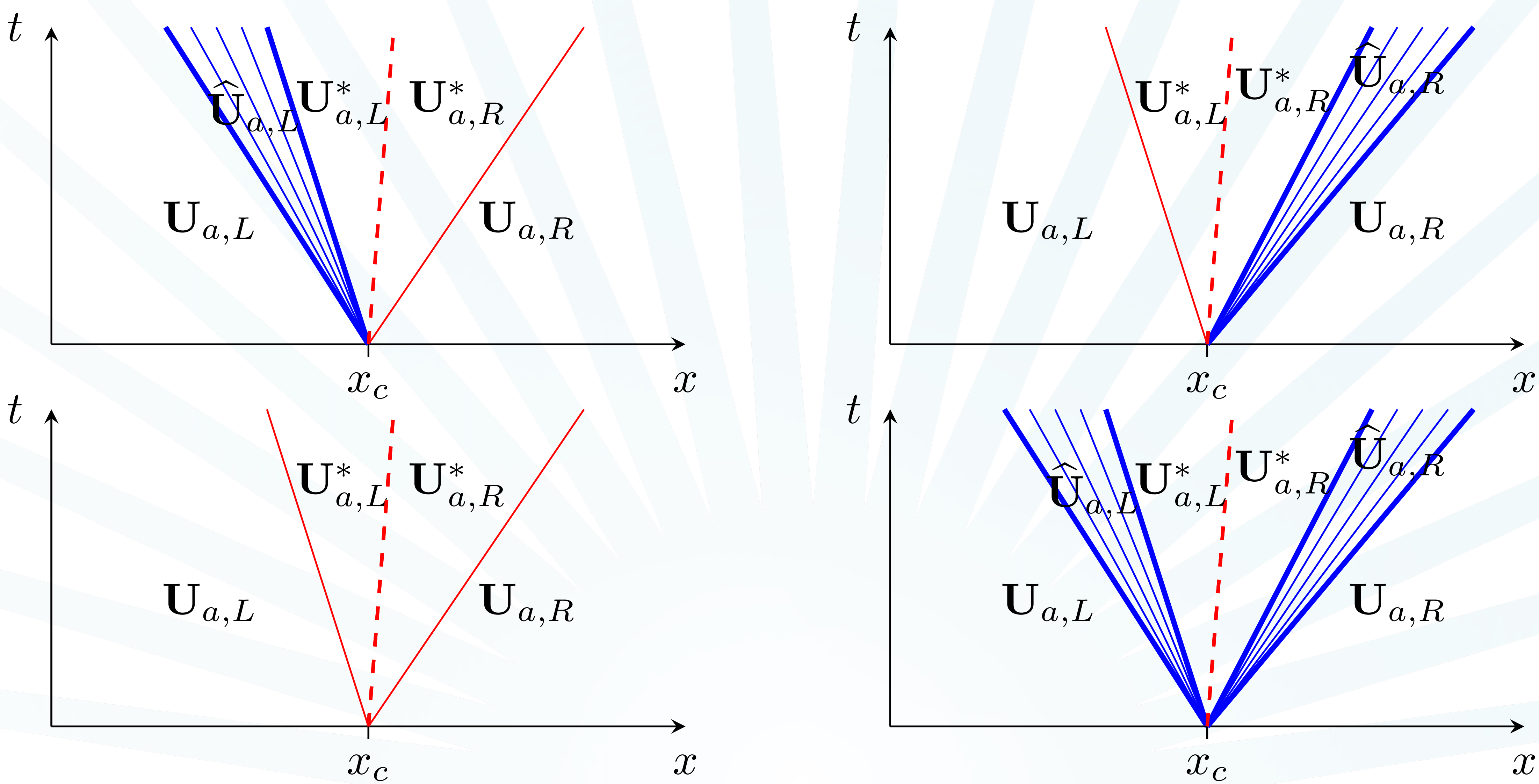

Classical Numerical Schemes

Classical numerical schemes

Godunov-type schemes

approximate Riemann solvers are used

Step 0 : initial data discretisation

Step 1 : solution of a Riemann problem for each interface $x_{j-1/2}$ obtaining $\mathbf{v}(x, t^{n+1})$

Step 2 : average $\mathbf{V}_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}(x, t^{n+1}) \mathrm{d}x$

Remark: the state and the sensitivity systems are not solved as a global system.

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \\ \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S}(\mathbf{U}) \end{cases} \qquad \mathbf{S}(\mathbf{U}) = \sum_{k=1}^{N_s} \sigma_{a,k}(\mathbf{U}_k^+ - \mathbf{U}_k^-)$$

Remark: HLL-type schemes cannot be used for the state, two intermediate star states are necessary to have a well-defined the source term for the sensitivity.

Approximate Riemann solver for the state

- First order Roe-type scheme

$$\lambda_1^{ROE} = \tilde{u} - \tilde{c} \qquad \lambda_2^{ROE} = \tilde{u} \qquad \lambda_3^{ROE} = \tilde{u} + \tilde{c} \qquad \text{Roe-averaged eigenvalues}$$

$$\mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{3} \alpha_i \tilde{\mathbf{r}}_i \qquad \text{decomposition along Roe-averaged eigenvectors}$$

$$\mathbf{U}_L^* = \mathbf{U}_L + \alpha_1 \tilde{\mathbf{r}}_1 \qquad \mathbf{U}_R^* = \mathbf{U}_R - \alpha_3 \tilde{\mathbf{r}}_3$$

[10] Bouchut, F. (2004). Nonlinear stability of finite Volume Methods for hyperbolic conservation laws: And Well-Balanced schemes for sources. Springer Science & Business Media.

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \\ \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S}(\mathbf{U}) \end{cases} \qquad \mathbf{S}(\mathbf{U}) = \sum_{k=1}^{N_s} \sigma_{a,k}(\mathbf{U}_k^+ - \mathbf{U}_k^-)$$

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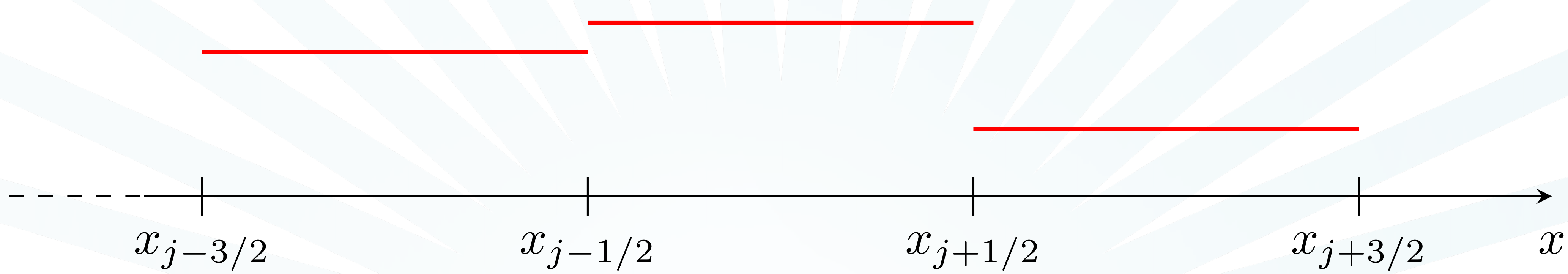
Approximate Riemann solver for the state

- First order Roe-type scheme
- Second order Roe-type scheme

  Time discretisation: two-step Runge-Kutta method

  Space discretisation: MUSCL-type scheme [10]



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Classical numerical schemes

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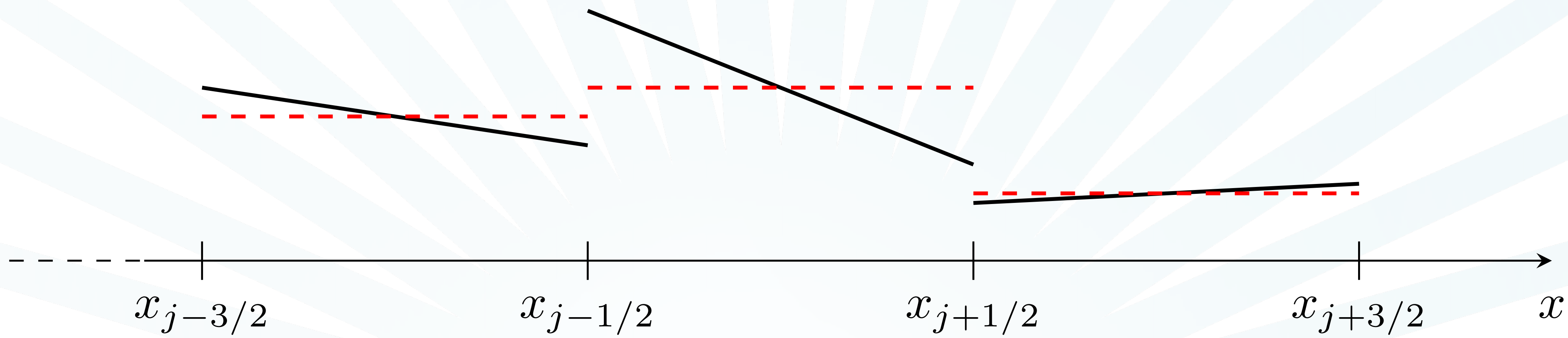

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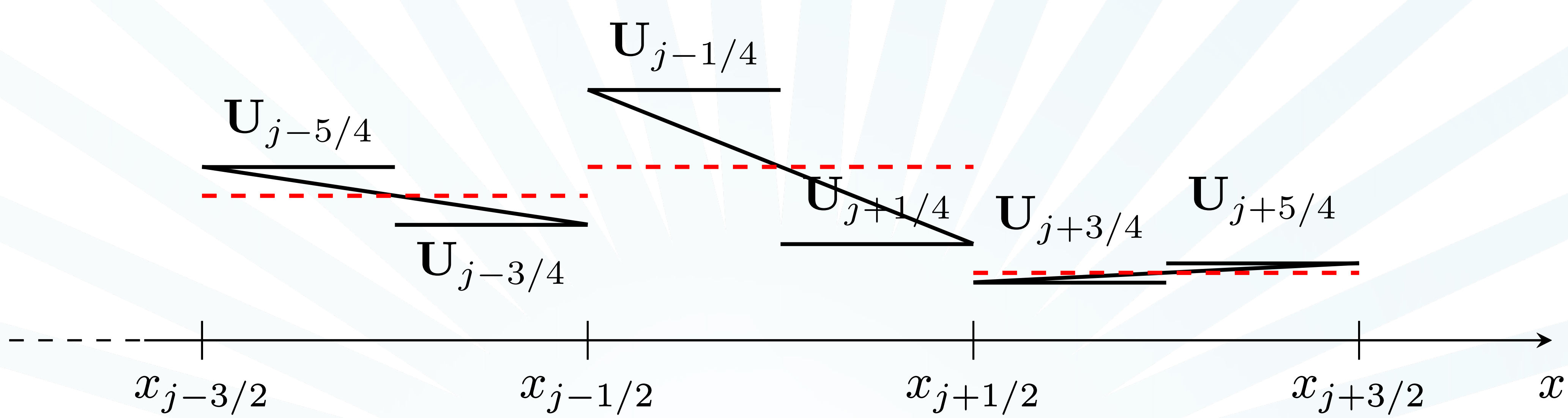

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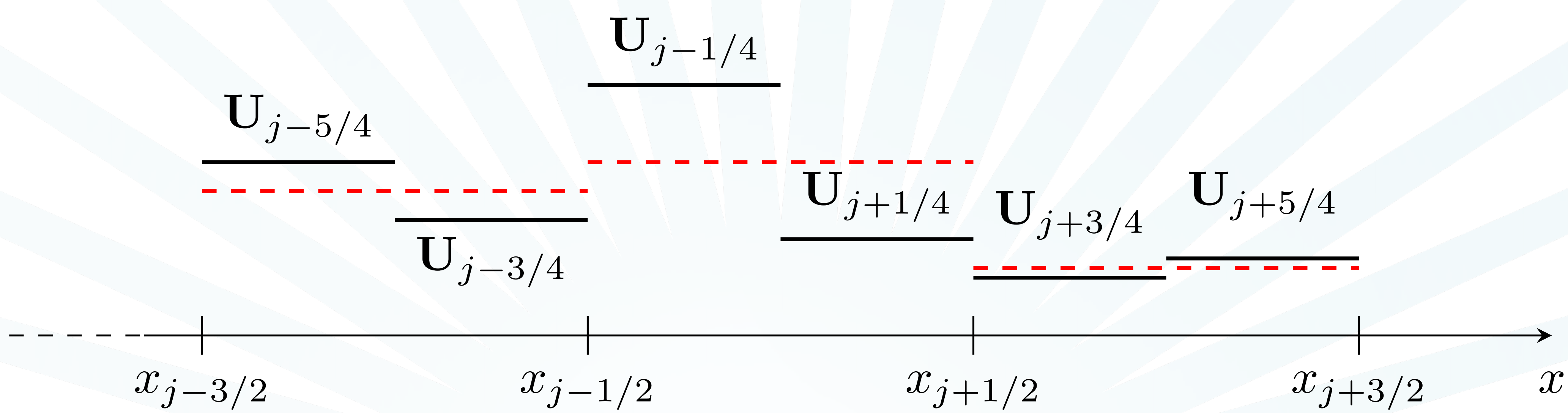

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Approximate Riemann solvers for the sensitivity

- HLL-type scheme: simpler structure that the state solver.

HLL consistency conditions yield:

$$\mathbf{U}^*_{a,j-1/2} = \frac{1}{\lambda_3^{ROE} - \lambda_1^{ROE}}\left(\lambda_3^{ROE}\mathbf{U}^n_{a,j} - \lambda_1^{ROE}\mathbf{U}^n_{a,j-1} - \mathbf{F}_a(\mathbf{U}_j, \mathbf{U}_{a,j}) + \mathbf{F}_a(\mathbf{U}_{j-1}, \mathbf{U}_{a,j-1}) + \mathbf{S}_{j-1/2}\right)$$

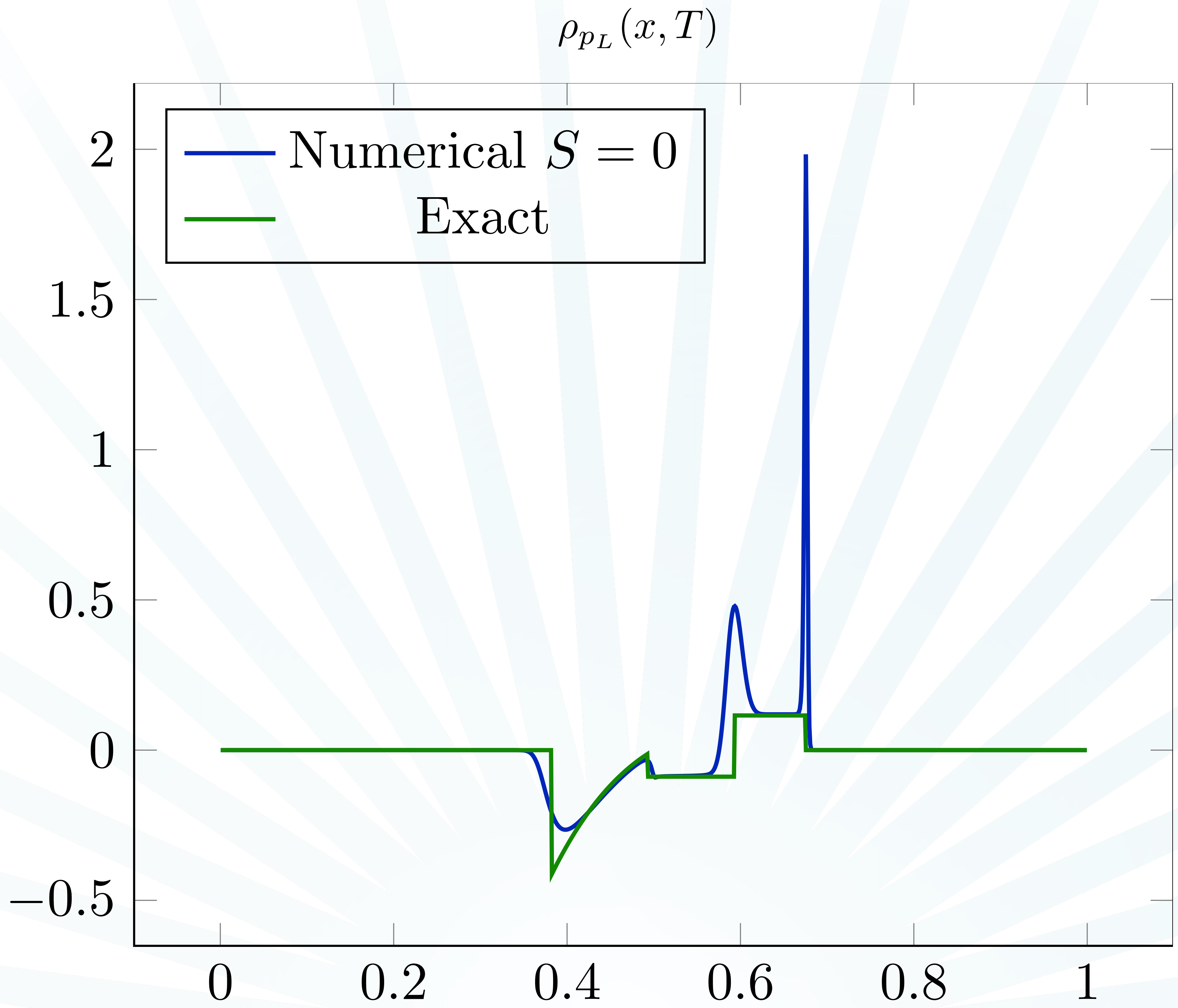
$$\begin{aligned}\mathbf{S}_{j-1/2} =& \partial_a\lambda^{ROE}_{1,j-1/2}(\mathbf{U}^*_{L,j-1/2} - \mathbf{U}_{j-1})d_{1,j-1/2}\\ &+\partial_a\lambda^{ROE}_{2,j-1/2}(\mathbf{U}^*_{R,j-1/2} - \mathbf{U}^*_{L,j-1/2})\\ &+\partial_a\lambda^{ROE}_{3,j-1/2}(\mathbf{U}_j - \mathbf{U}^*_{R,j-1/2})d_{3,j-1/2}\end{aligned}$$

- HLLC-type scheme: same structure as the state.

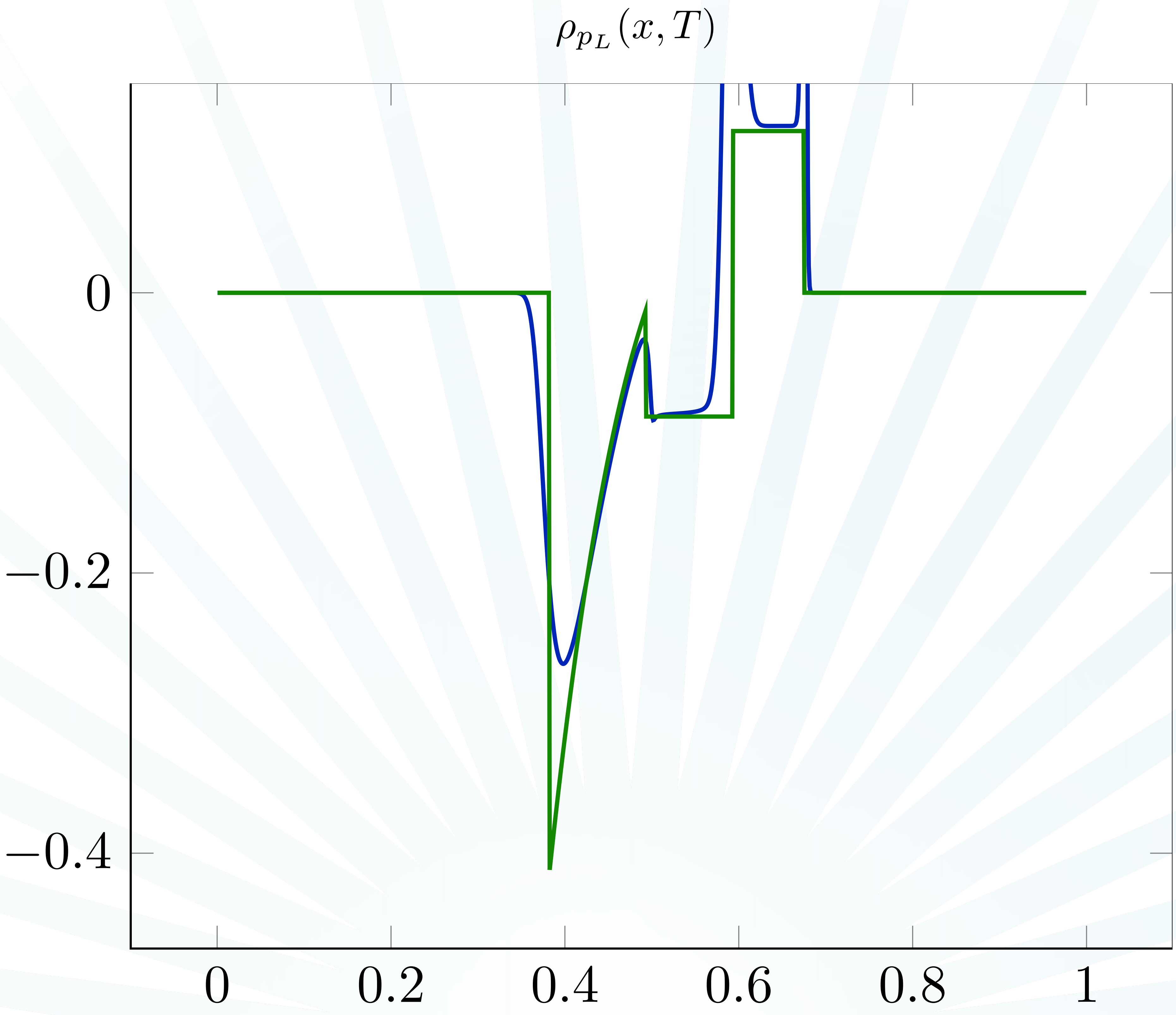
HLL consistency conditions + Rankine-Hugoniot conditions. Equivalent to:

$$\mathbf{U}^*_{a,L} = \mathbf{U}_{a,L} + \alpha_{1,a}\tilde{\mathbf{r}}_1 + \alpha_1\tilde{\mathbf{r}}_{1,a} \qquad \mathbf{U}^*_{a,R} = \mathbf{U}_{a,R} - \alpha_{3,a}\tilde{\mathbf{r}}_3 - \alpha_3\tilde{\mathbf{r}}_{3,a}$$

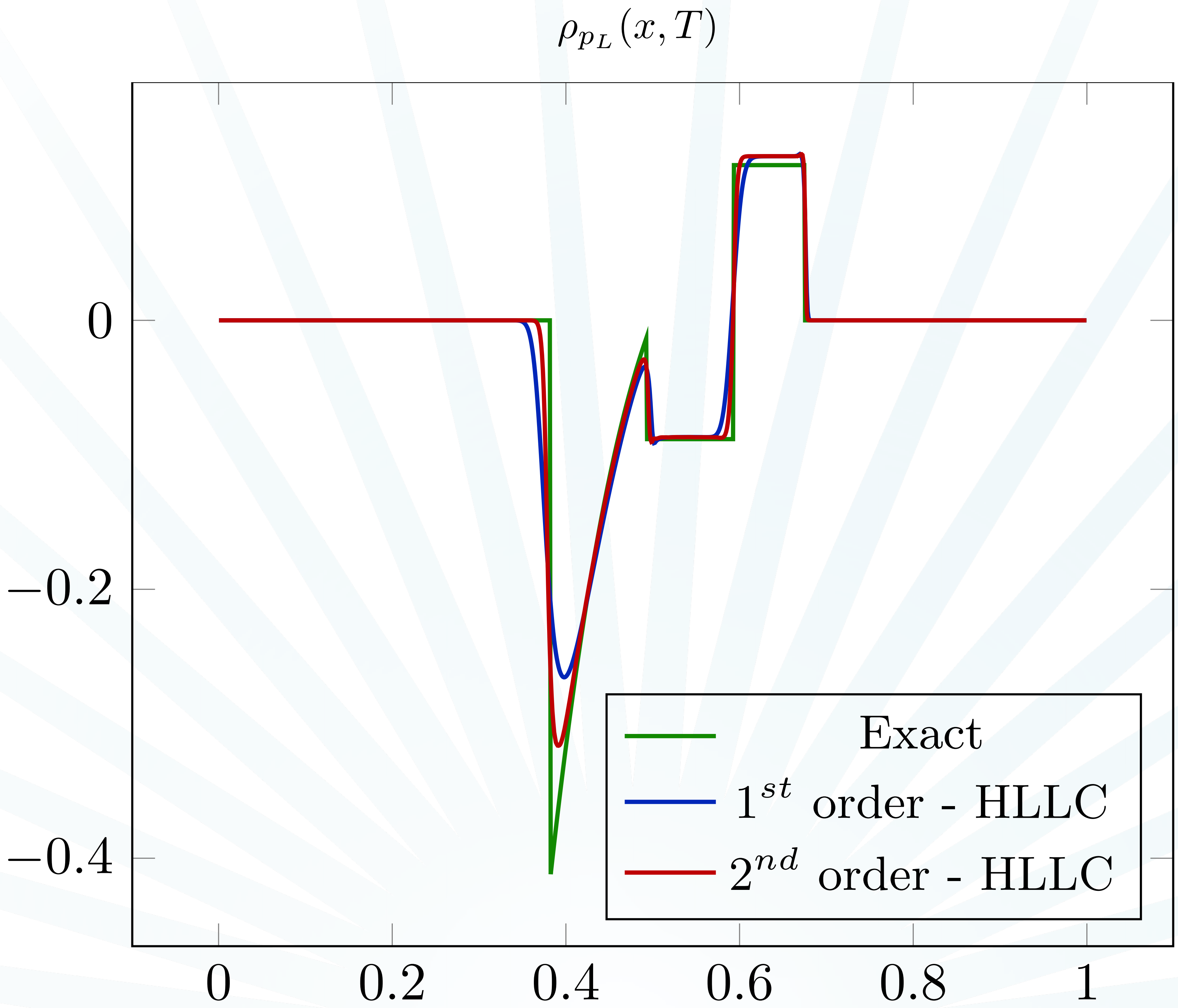
Classical numerical schemes



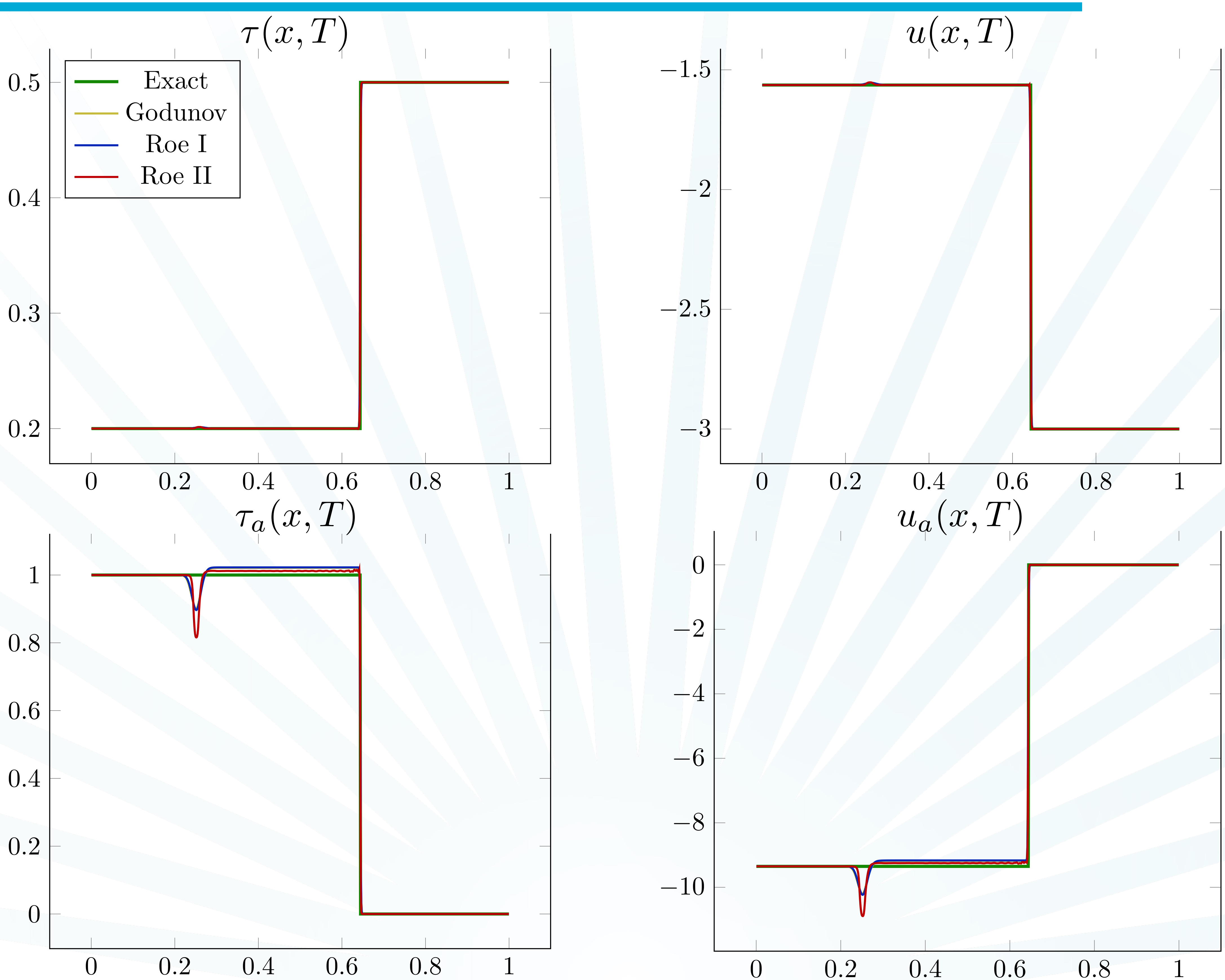
Classical numerical schemes



Classical numerical schemes



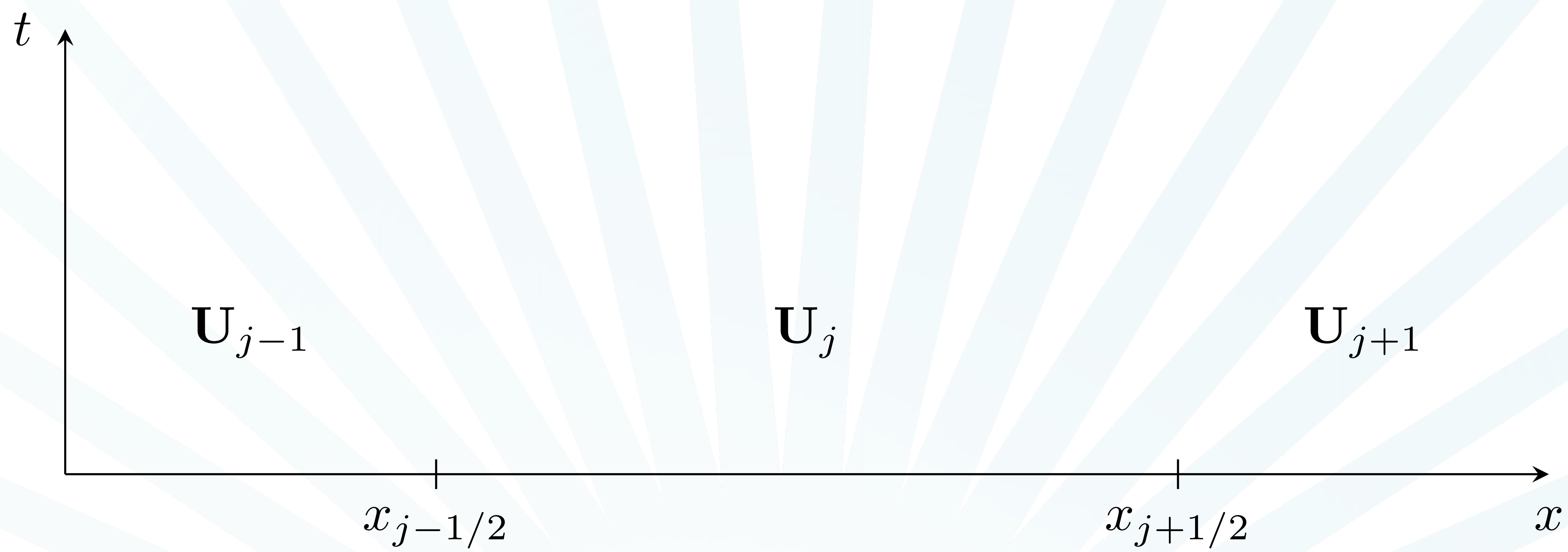
Isolated shock for the p-system



Numerical Scheme without diffusion

Scheme without numerical diffusion

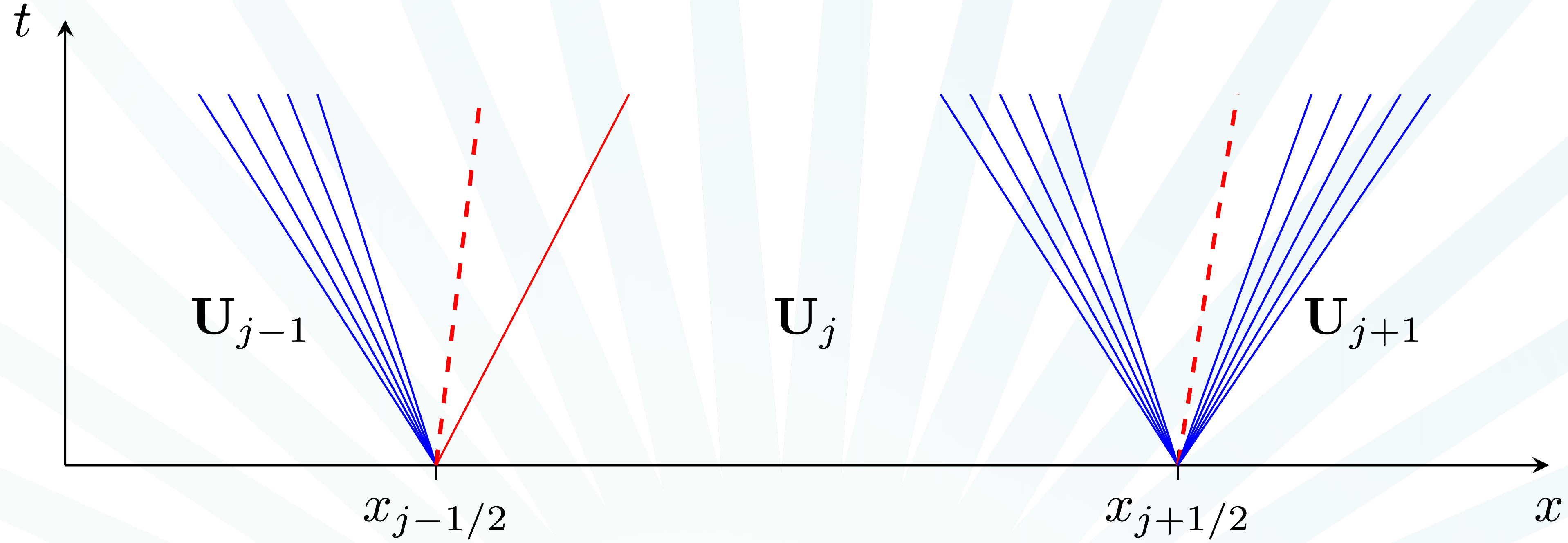
Step 0 : initial data discretisation



Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

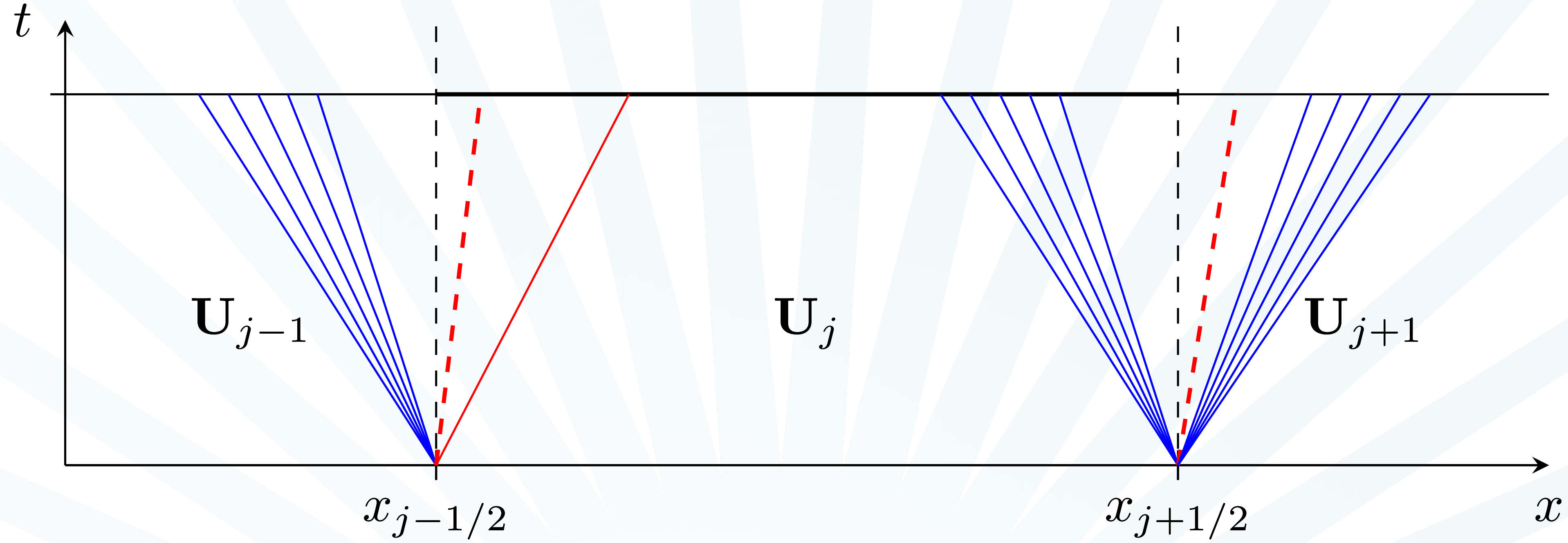


Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : average



Scheme without numerical diffusion

Step 0 : initial data discretisation

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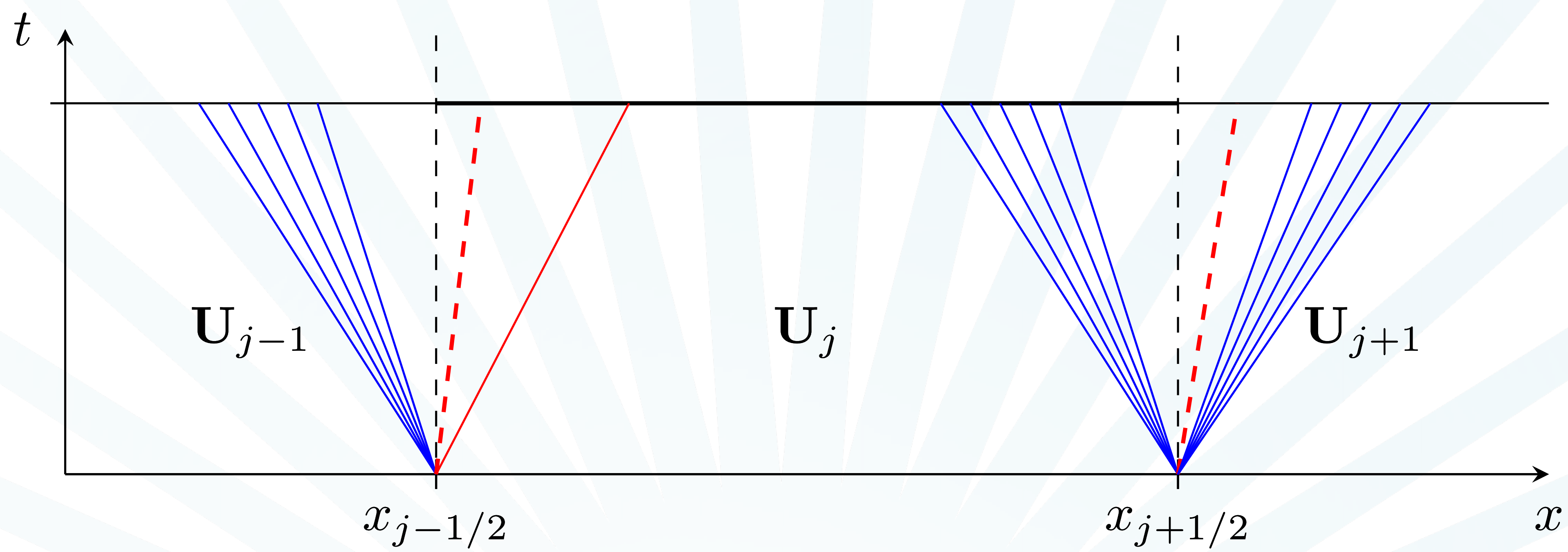
Step 2 : ~~average~~

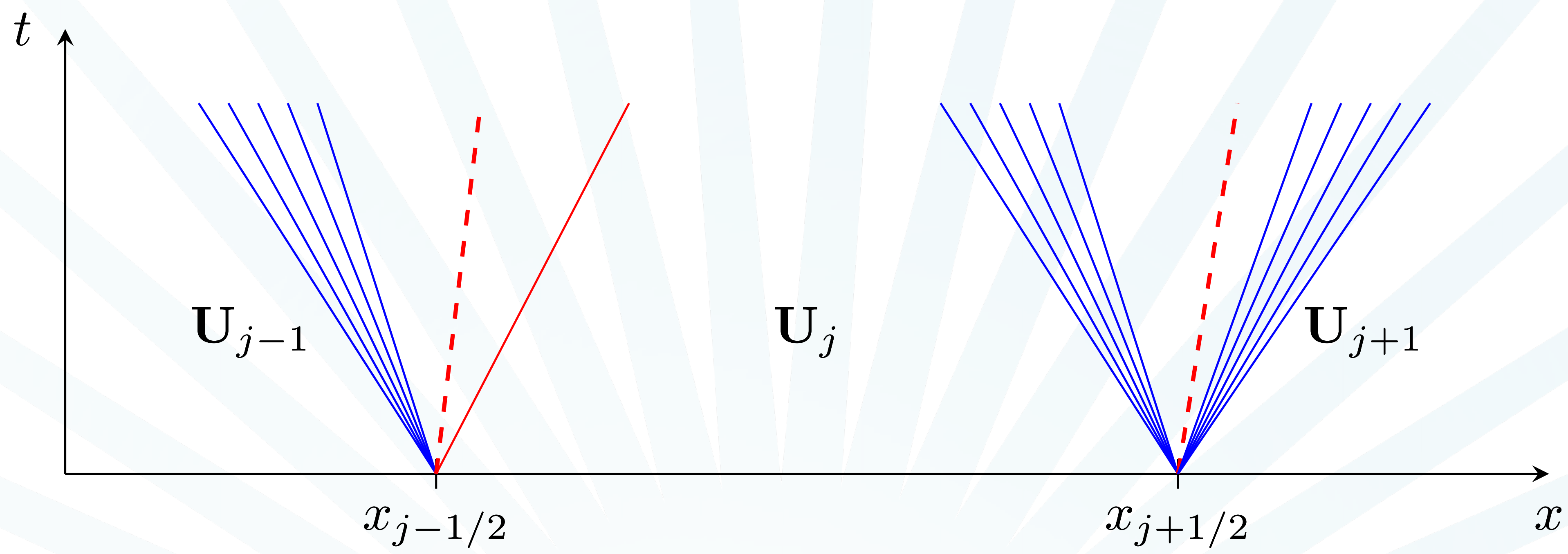


Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed [11]



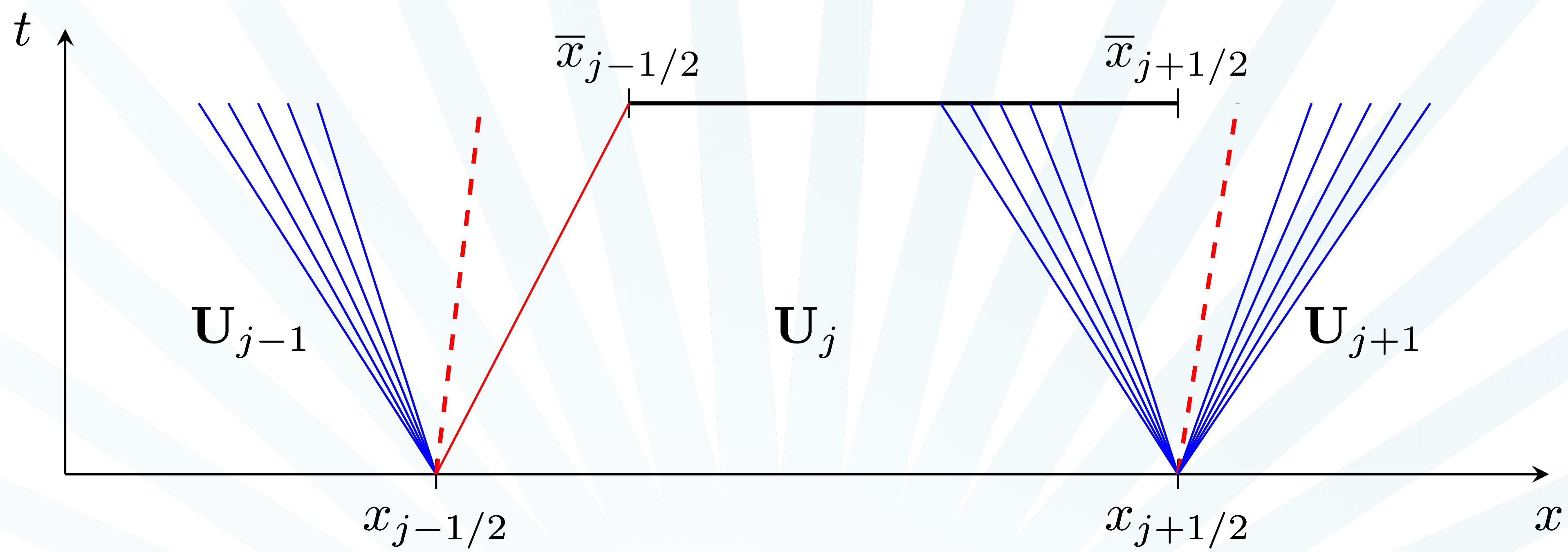
[11] Chalons, C., & Goatin, P. (2008). Godunov scheme and sampling technique for computing phase transitions in traffic flow modeling. *Interfaces and Free Boundaries*, *10*(2), 197-221.

Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed [11]



$$\overline{x}_{j-1/2} = x_{j-1/2} + \sigma_{j-1/2}\Delta t$$

[11] Chalons, C., & Goatin, P. (2008). Godunov scheme and sampling technique for computing phase transitions in traffic flow modeling. *Interfaces and Free Boundaries*, *10*(2), 197-221.

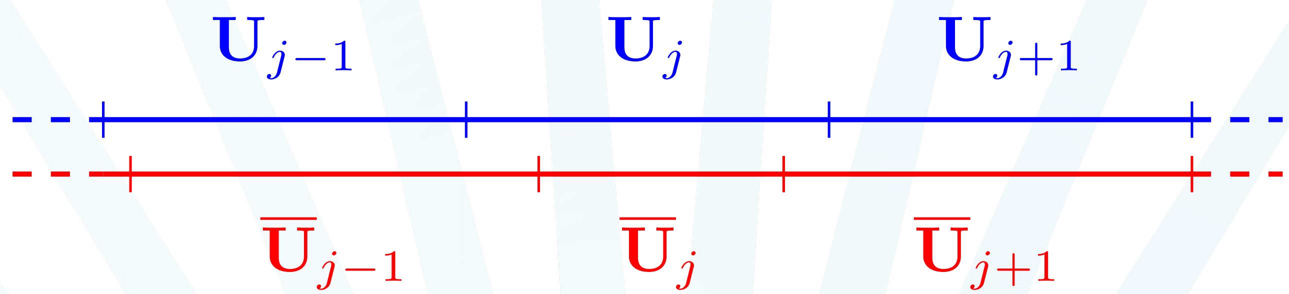
Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed

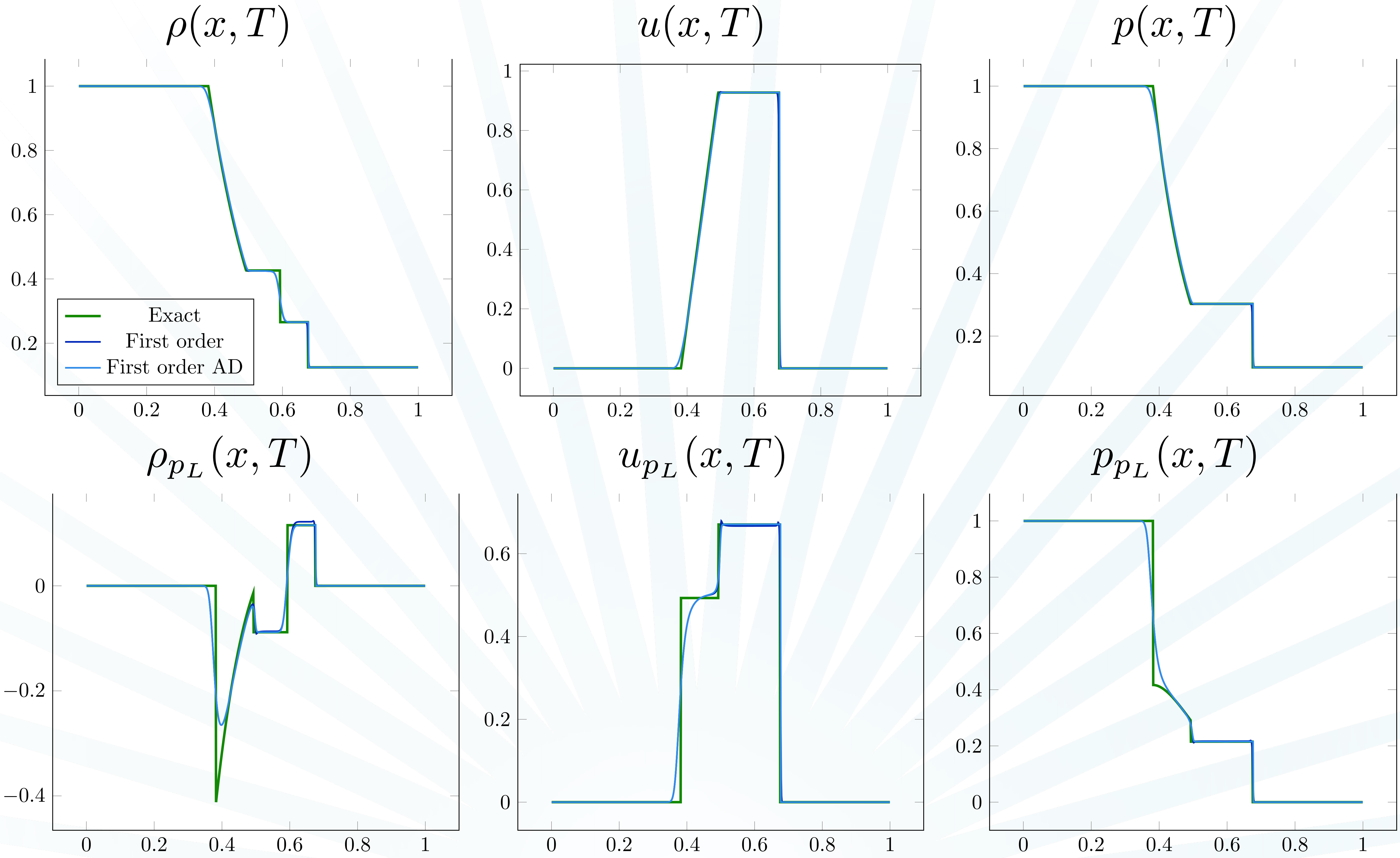
Step 3 : projection on the initial mesh [12]



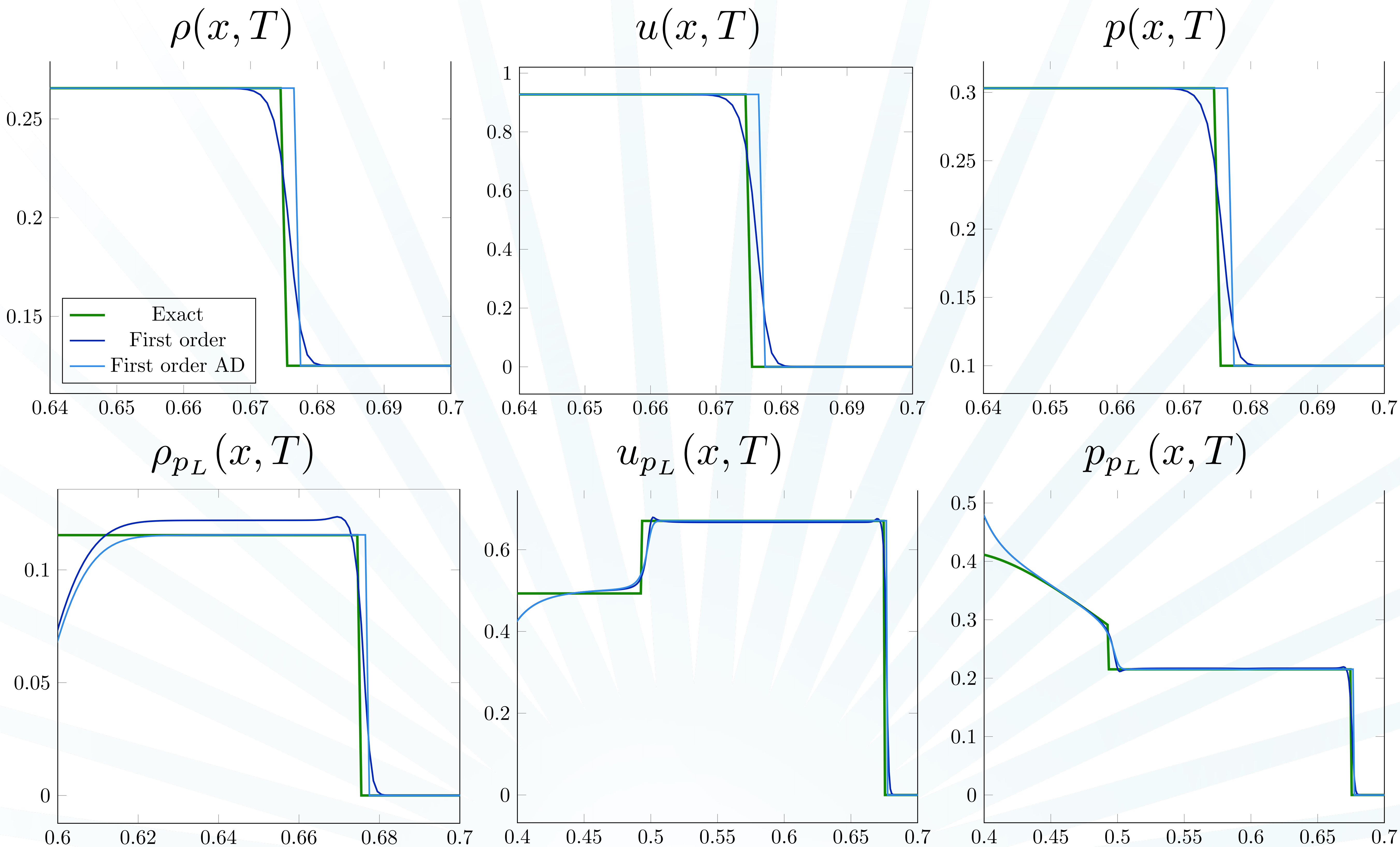
$$\mathbf{U}_j = \begin{cases} \overline{\mathbf{U}}_{j-1} & \text{if } \alpha \in \left(0, \frac{\Delta t}{\Delta x} \max(\sigma_{j-1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j} & \text{if } \alpha \in \left[\frac{\Delta t}{\Delta x} \max(\sigma_{j-1/2}, 0), 1 + \frac{\Delta t}{\Delta x} \min(\sigma_{j+1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j+1} & \text{if } \alpha \in \left[1 + \frac{\Delta t}{\Delta x} \min(\sigma_{j+1/2}, 0), 1\right). \end{cases}$$

$$\alpha \sim \mathcal{U}([0,1])$$

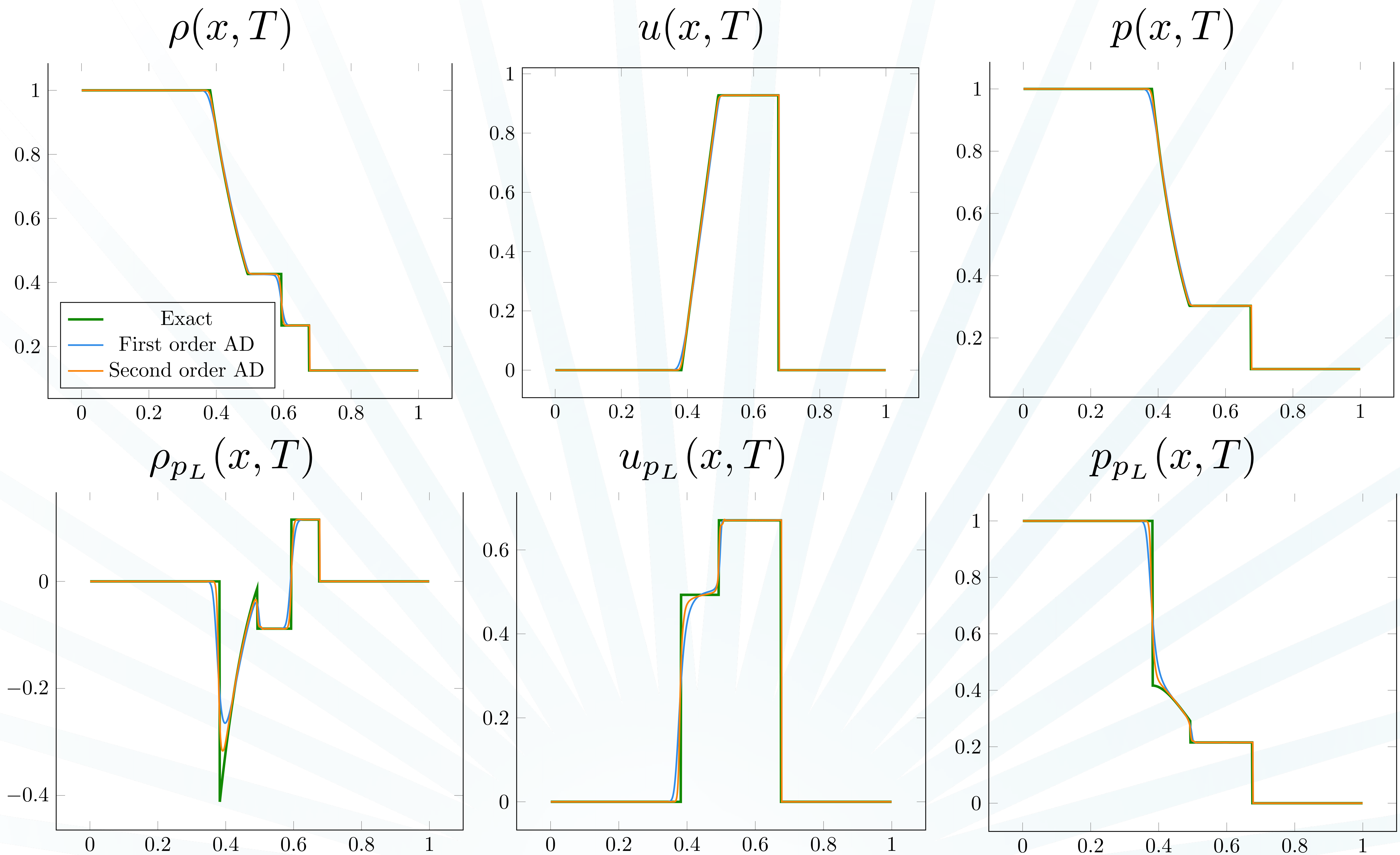
[12] Glimm, J. (1965). Solutions in the large for nonlinear hyperbolic systems of equations. *Communications on pure and applied mathematics*, *18*(4), 697-715.


$\rho(x,T)$
$u(x,T)$
$p(x,T)$
$\rho_{p_L}(x,T)$
$u_{p_L}(x,T)$
$p_{p_L}(x,T)$
Exact
First order
First order AD

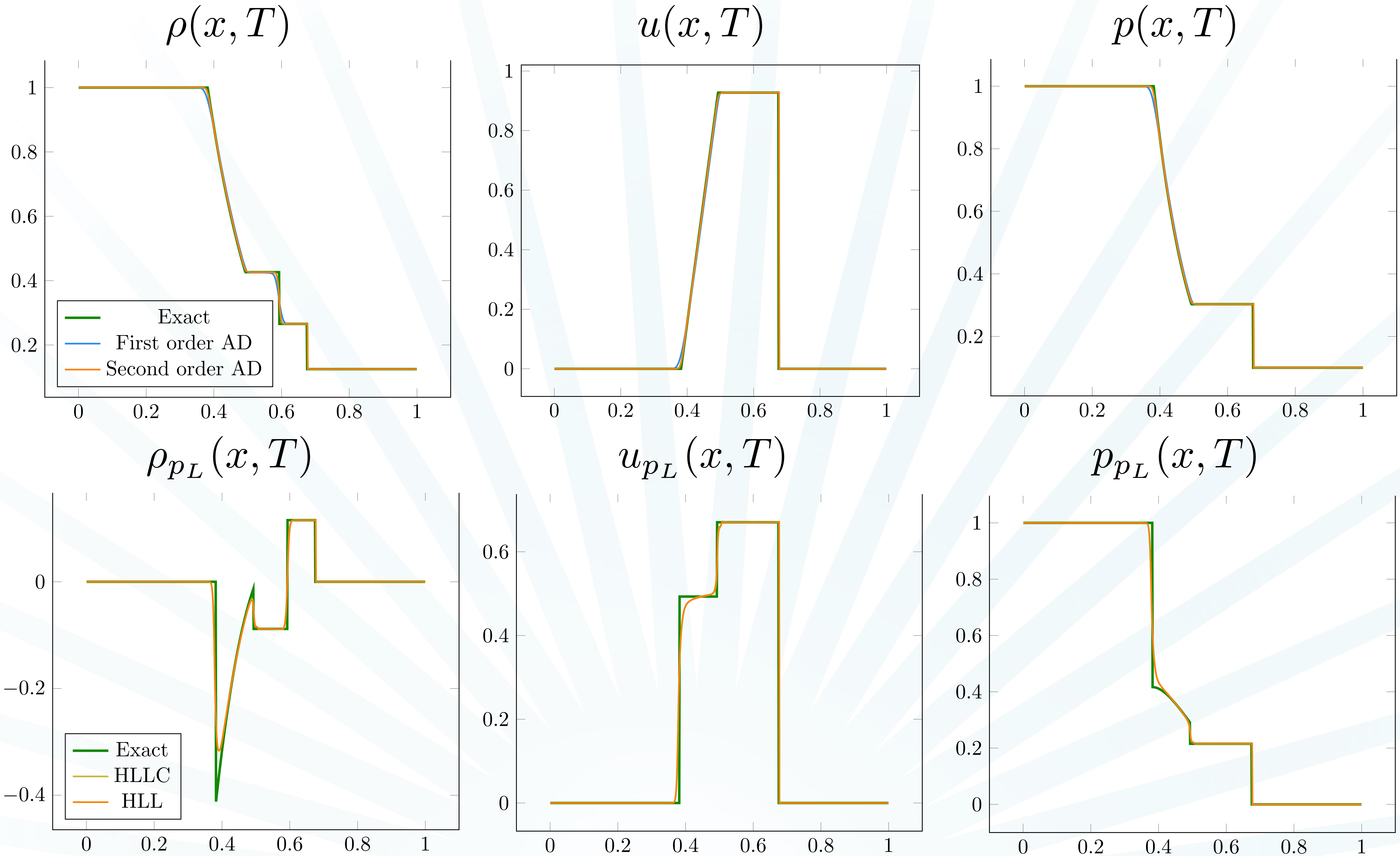
Numerical results



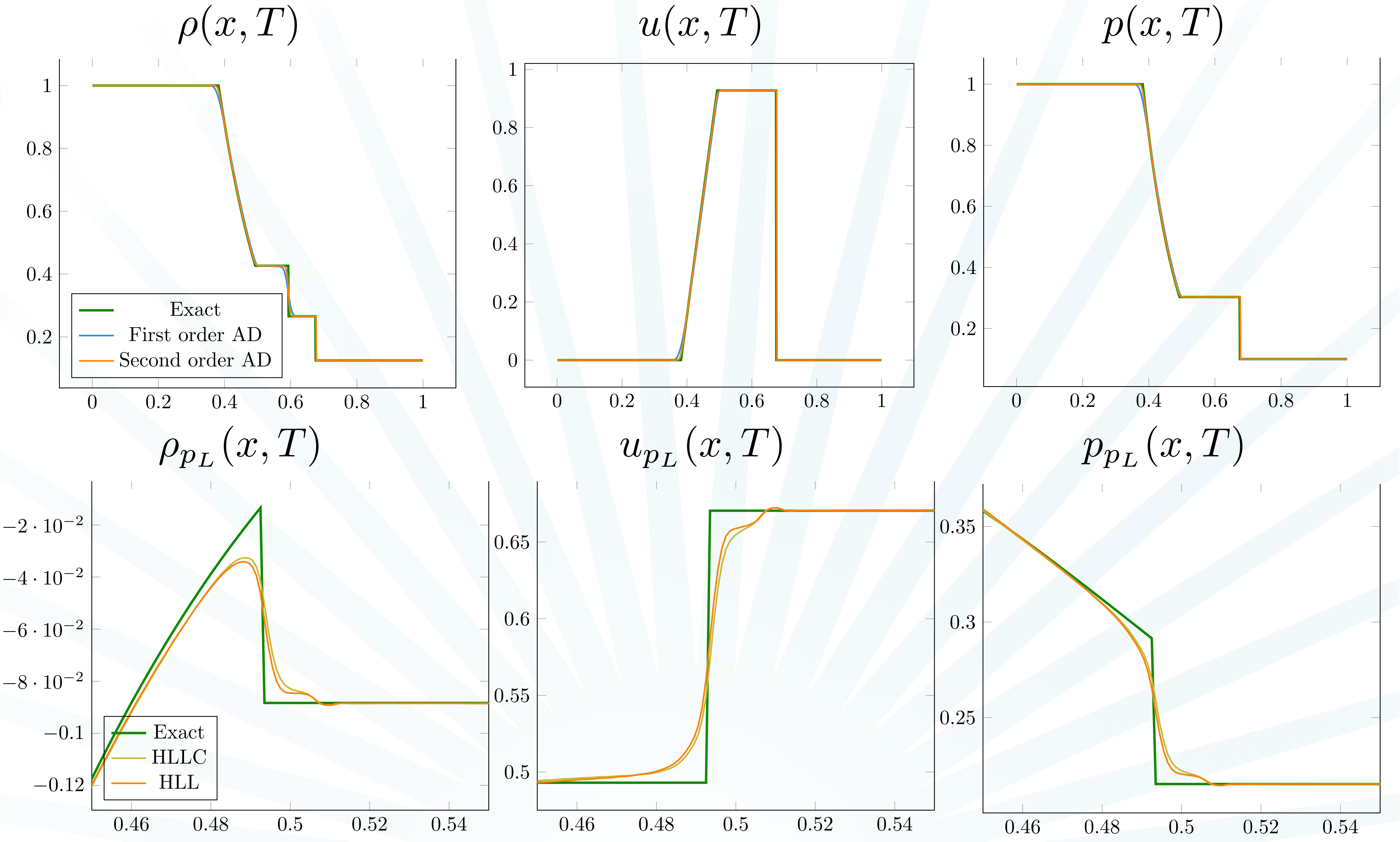
Numerical results



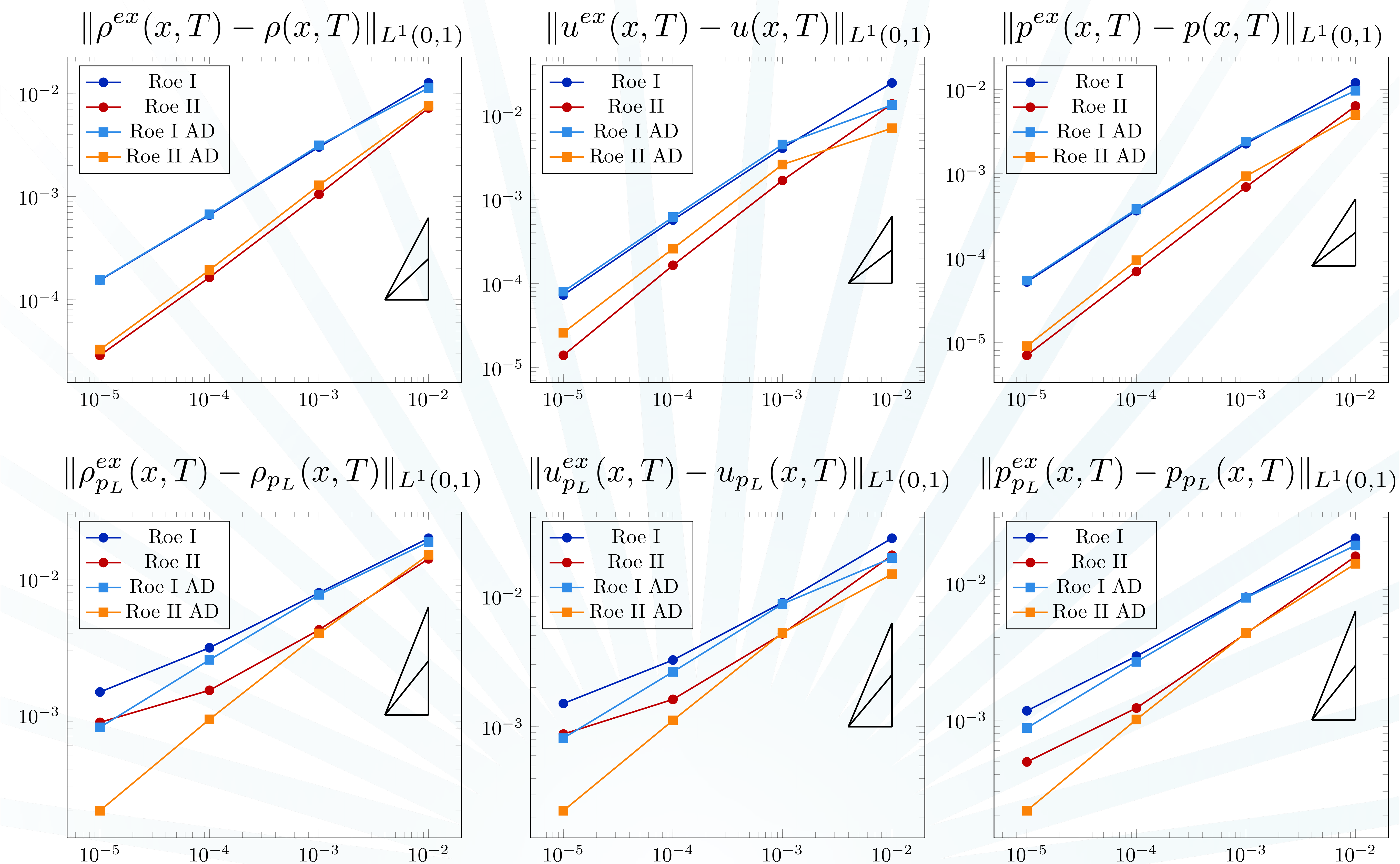
Numerical results



Numerical results



Convergence



Applications

Let **a** be a random vector, with the following average and variance:

$$\mu_{\mathbf{a}} = \begin{bmatrix} \mu_{a_1} \\ \vdots \\ \mu_{a_M} \end{bmatrix}, \qquad \sigma_{\mathbf{a}} = \begin{bmatrix} \sigma^2_{a_1} & \mathrm{cov}(a_1, a_2) & \dots & \mathrm{cov}(a_1, a_M) \\ \mathrm{cov}(a_1, a_2) & \sigma^2_{a_2} & \dots & \mathrm{cov}(a_2, a_M) \\ \vdots & & \ddots & \vdots \\ \mathrm{cov}(a_1, a_M) & \dots & & \sigma^2_{a_M} \end{bmatrix},$$

Aim: determine a **confidence interval** $CI_X = [\mu_X - \kappa\sigma_X, \mu_X + \kappa\sigma_X]$

Monte Carlo approach: *N* samples of the state X_k

$$\mu_X = \frac{1}{N}\sum_{k=1}^{N} X_k \qquad \sigma^2_X = \frac{1}{N-1}\sum_{k=1}^{N} |\mu_X - X_k|^2$$

Sensitivity approach: let us consider the following first order Taylor expansion

$$X(\mathbf{a}) = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M} (a_i - \mu_{a_i}) X_{a_i}(\mu_{\mathbf{a}}) + o(\|\mathbf{a}\|^2).$$

Then, computing, the average one has:

$$\mu_X = E[X(\mathbf{a})] = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}}) E[a_i - \mu_{a_i}] = X(\mu_{\mathbf{a}}),$$

And for the variance:

$$\sigma_X^2 = E[(X(\mathbf{a}) - \mu_X)^2] = E\left[\left(\sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}})(a_i - \mu_{a_i})\right)^2\right] =$$

$$= \sum_{i=1}^{M} X_{a_i}^2(\mu_{\mathbf{a}}) E[(a_i - \mu_{a_i})^2] + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i}(\mu_{\mathbf{a}}) X_{a_j}(\mu_{\mathbf{a}}) E[(a_i - \mu_{a_i})(a_j - \mu_{a_j})].$$

Therefore, we have the following first order estimates:

$$\mu_X = X(\mu_{\mathbf{a}}), \quad \sigma_X^2 = \sum_{i=1}^{M} X_{a_i}^2 \sigma_{a_i}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i} X_{a_j} \operatorname{cov}(a_i, a_j).$$

Test case:

Riemann problem with uncertain parameters: $\mathbf{a} = (\rho_L, \rho_R, u_L, u_R, p_L, p_R)^t$

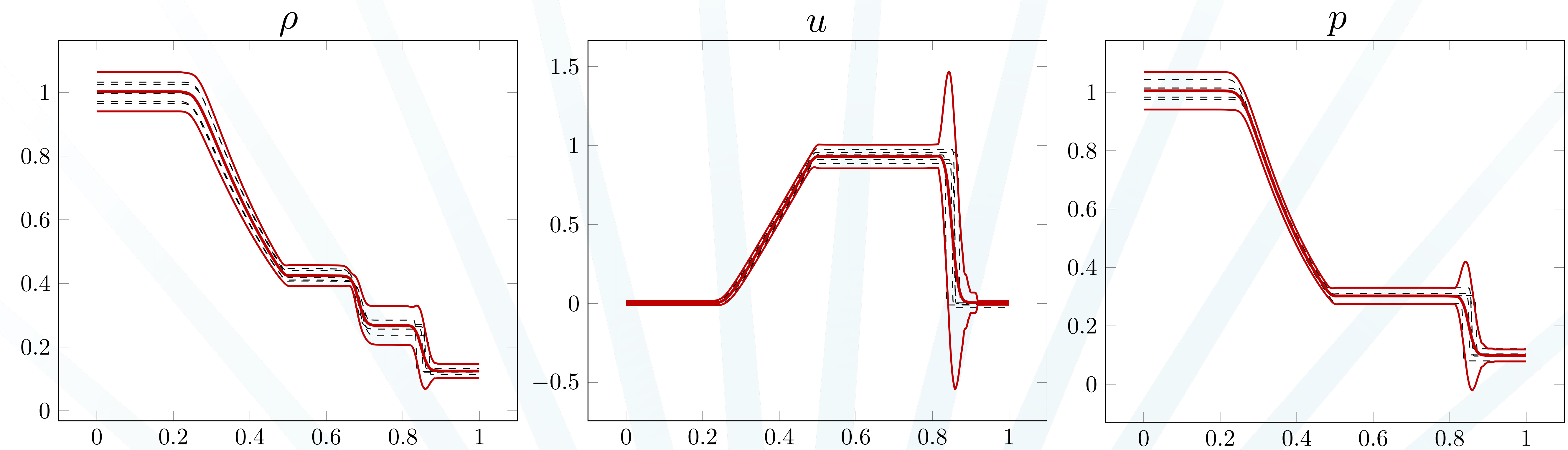
with the following average and covariance matrix:

$$\mu_{\mathbf{a}} = (1, 0.125, 0, 0, 1, 0.1)^t, \quad \sigma_{\mathbf{a}} = \mathrm{diag}(0.001, 0.000125, 0.0001, 0.0001, 0.001, 0.0001).$$

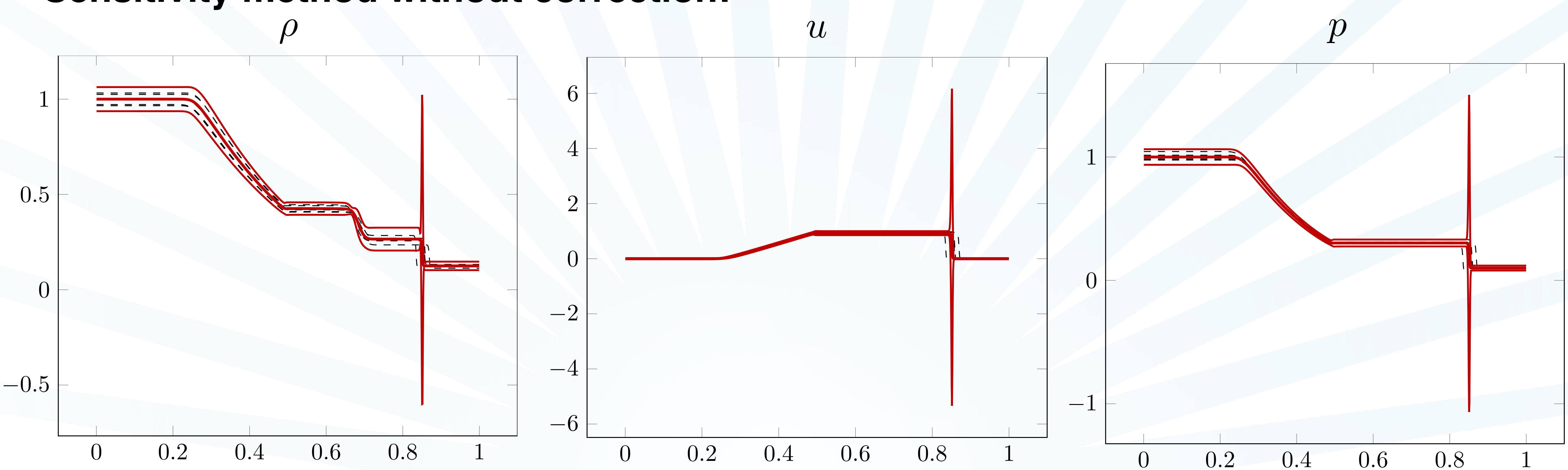
Since the covariance matrix is diagonal, the previous estimate is simplified:

$$\sigma_X^2 = \sum_{i=1}^{6} X_{a_i}^2 \sigma_{a_i}^2$$

Monte Carlo method:

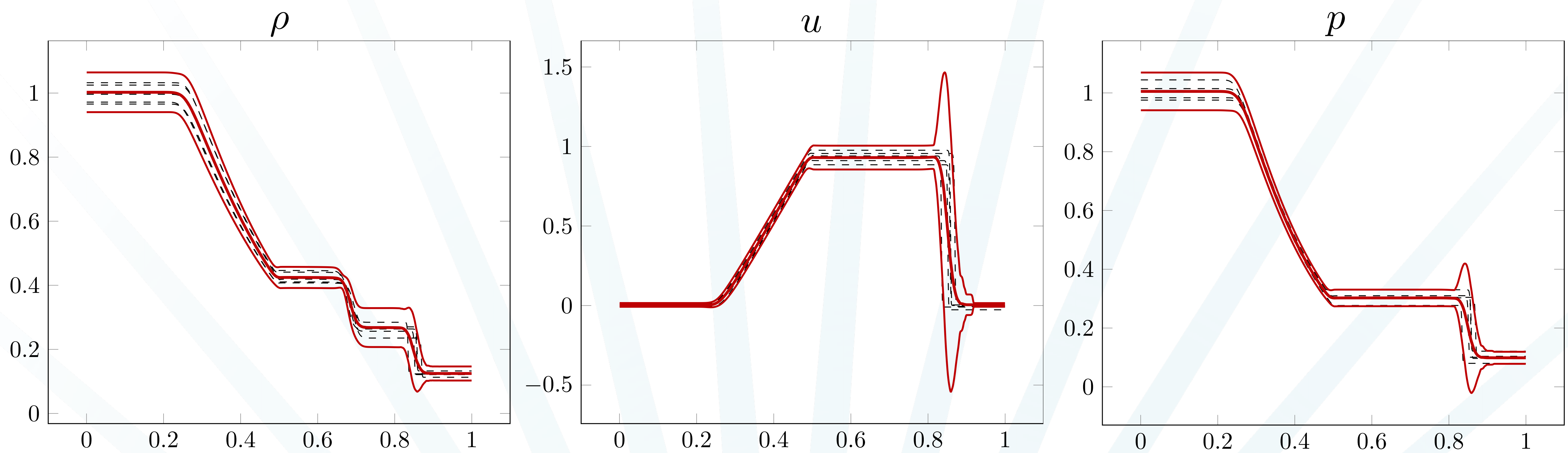


Sensitivity method without correction:

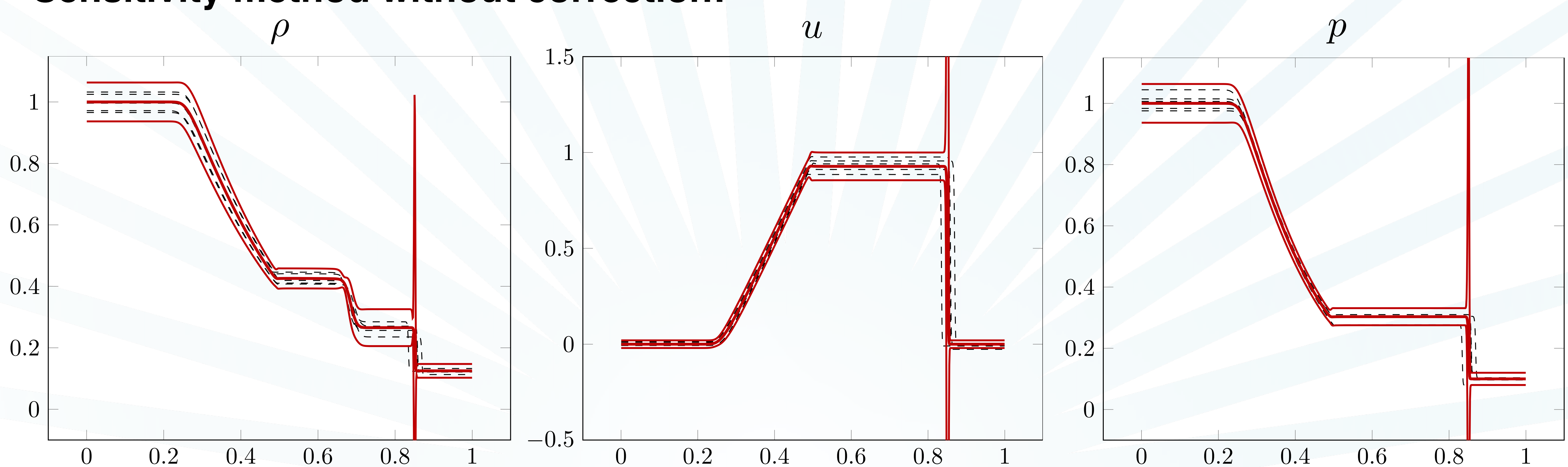


Uncertainty Quantification

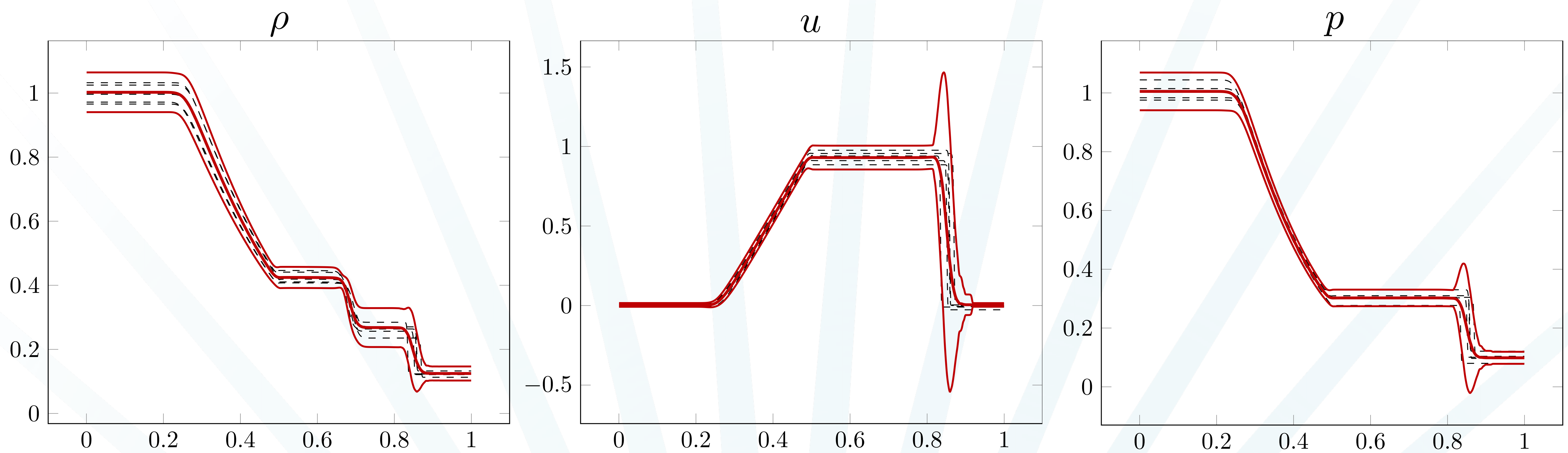
Monte Carlo method:



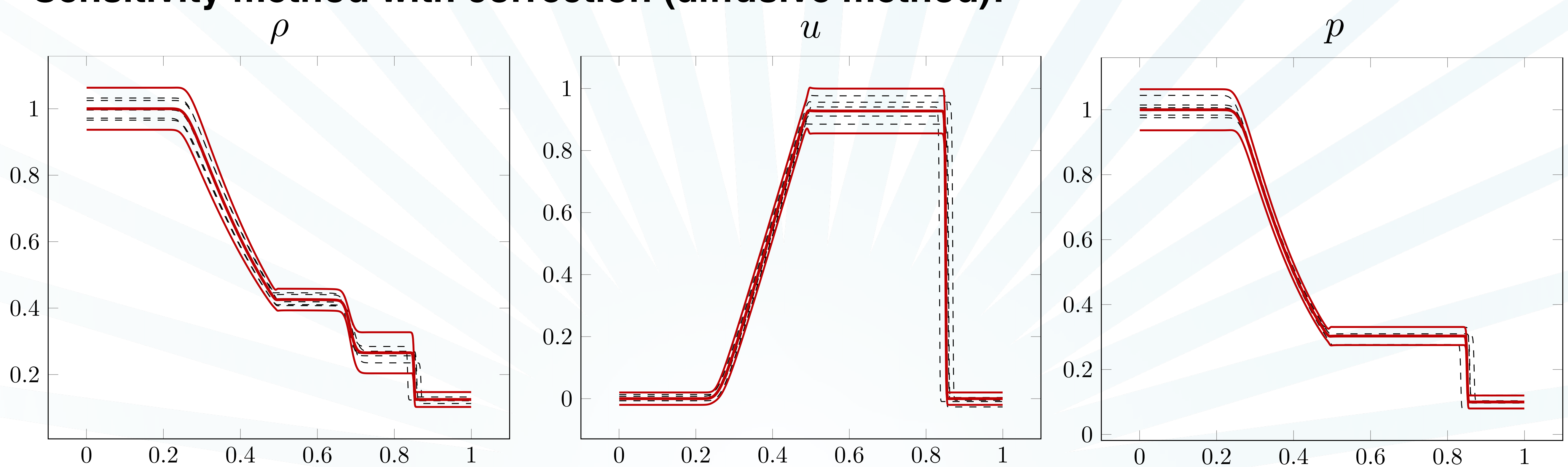
Sensitivity method without correction:



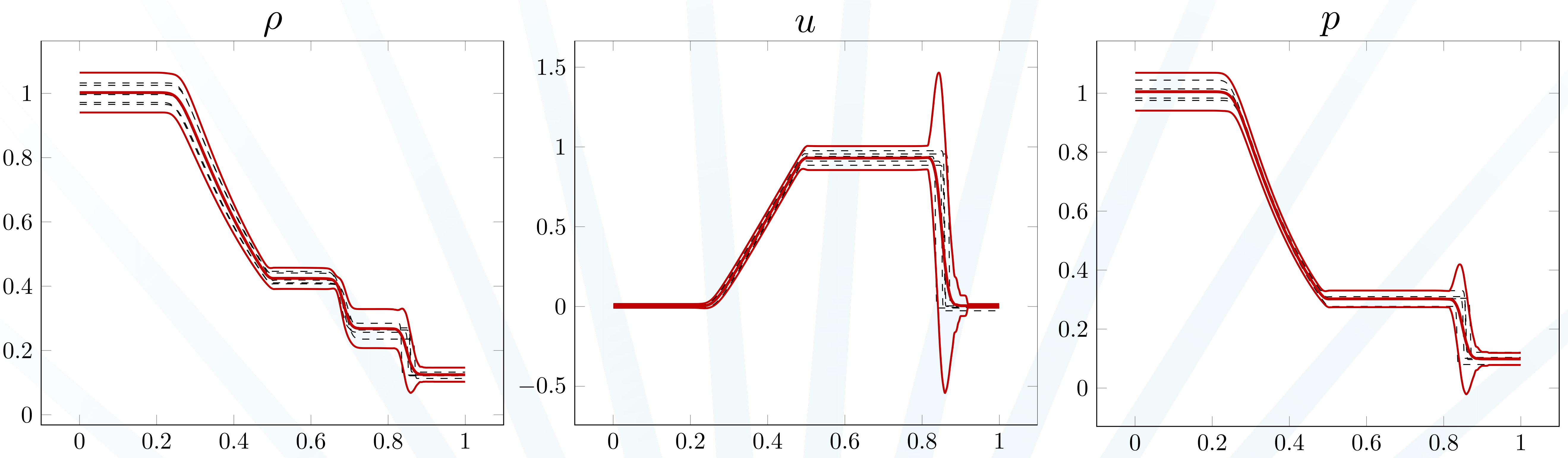
Monte Carlo method:



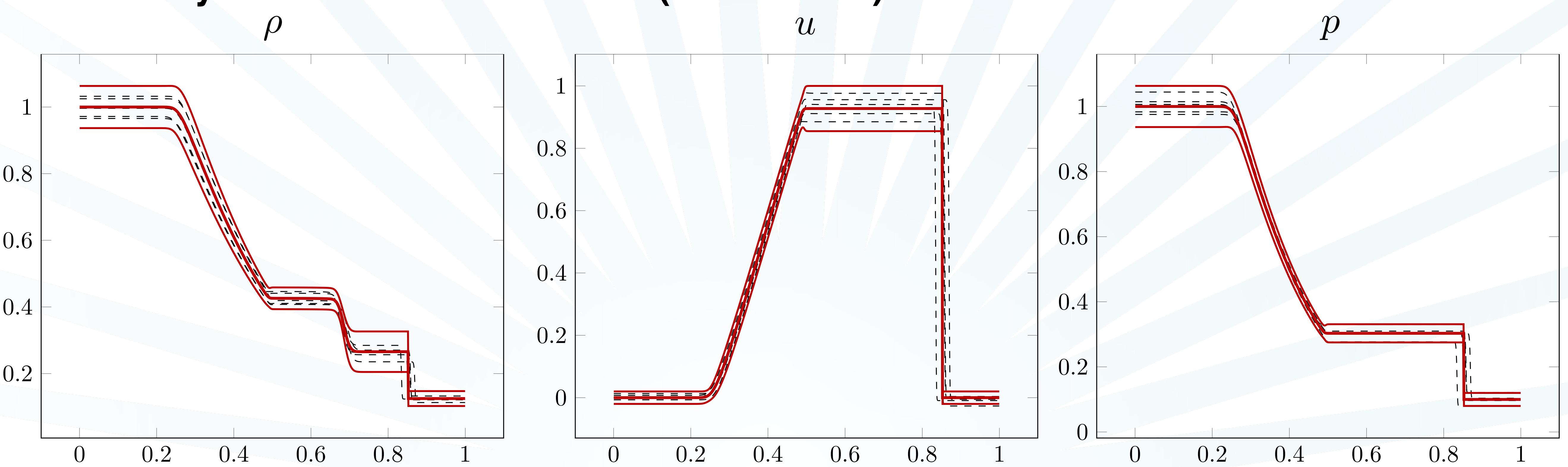
Sensitivity method with correction (diffusive method):



Monte Carlo method:



Sensitivity method with correction (AD method):

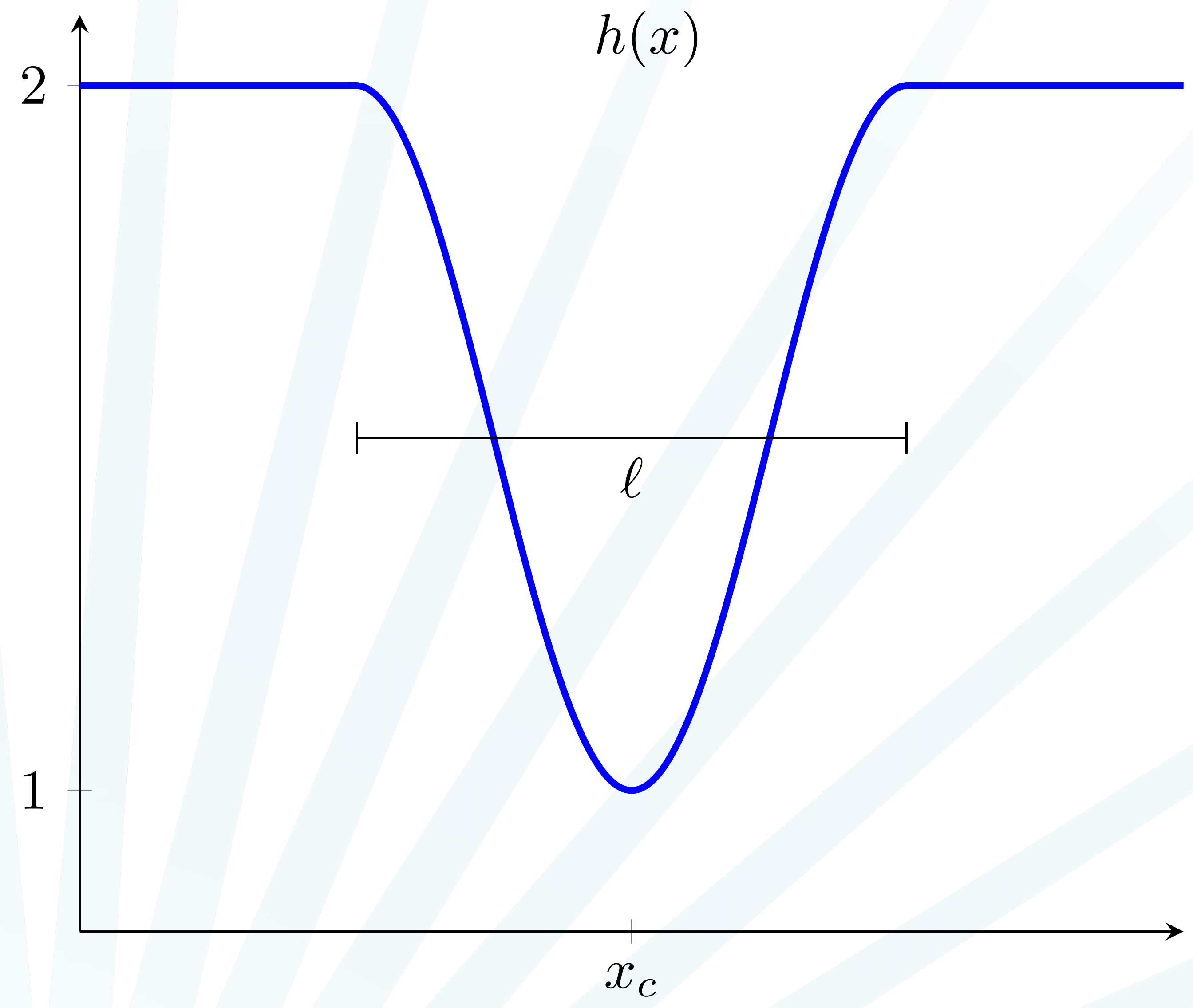


Optimization

The quasi-1D Euler equations are:

$$(1)\quad \begin{cases} \partial_t(h\rho) + \partial_x(h\rho u) = 0, \\ \partial_t(h\rho u) + \partial_x(h\rho u^2 + p) = p\partial_x h, \\ \partial_t(h\rho E) + \partial_x(hu(\rho E + p)) = 0, \end{cases}$$

+b.c.



Cost functional: $J(\mathbf{U}) = \frac{1}{2}\|p - p^*\|^2_{L^2}$

Parameters: $\mathbf{a} = (x_c, \ell)^t$

Target pressure: $p^* = p(x_c = 0.5, \ell = 0.5)$

Gradient: $\nabla_{\mathbf{a}} J(\mathbf{U}) = \begin{bmatrix} (p - p^*, p_{x_c})_{L^2} \\ (p - p^*, p_\ell)_{L^2} \end{bmatrix}$

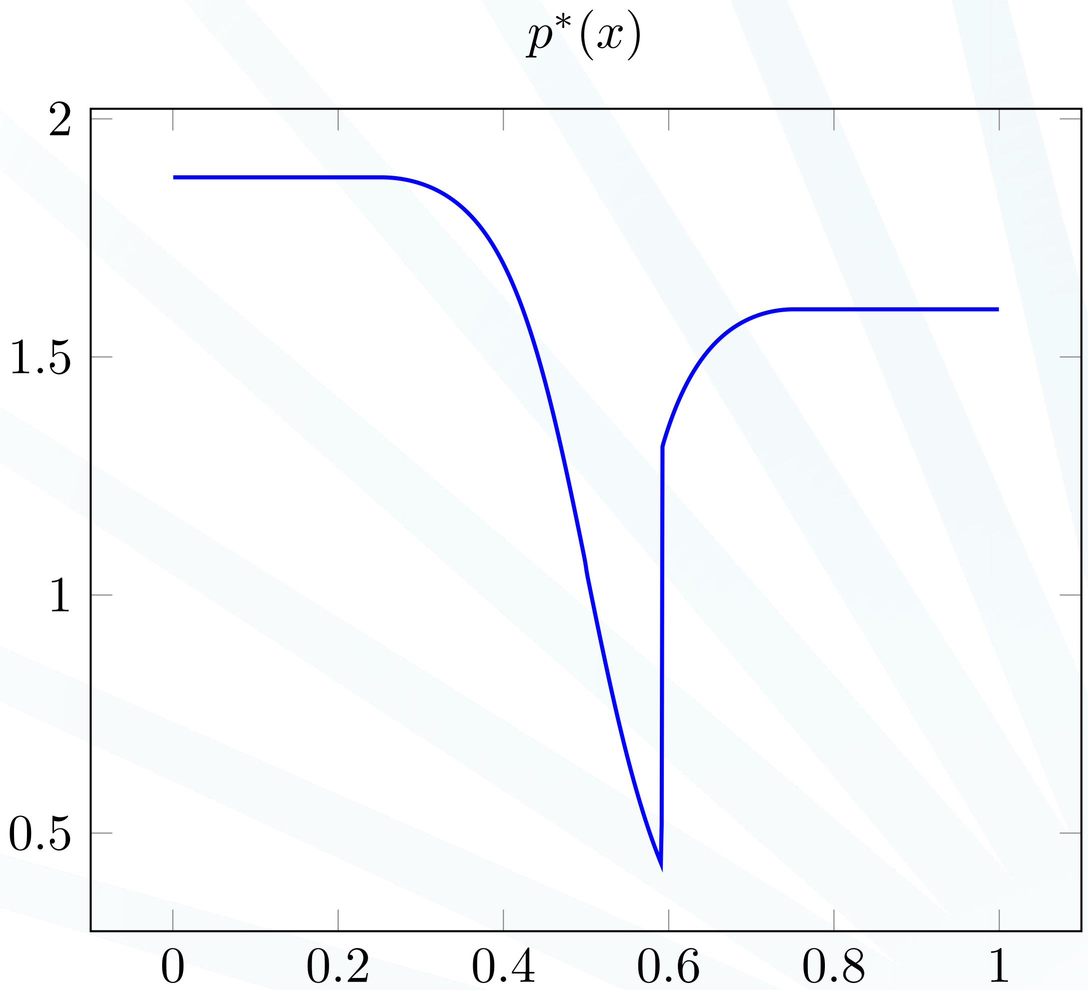
Optimization problem: $\min_{\mathbf{a}\in\mathcal{A}} J(\mathbf{U})$ subject to (1).

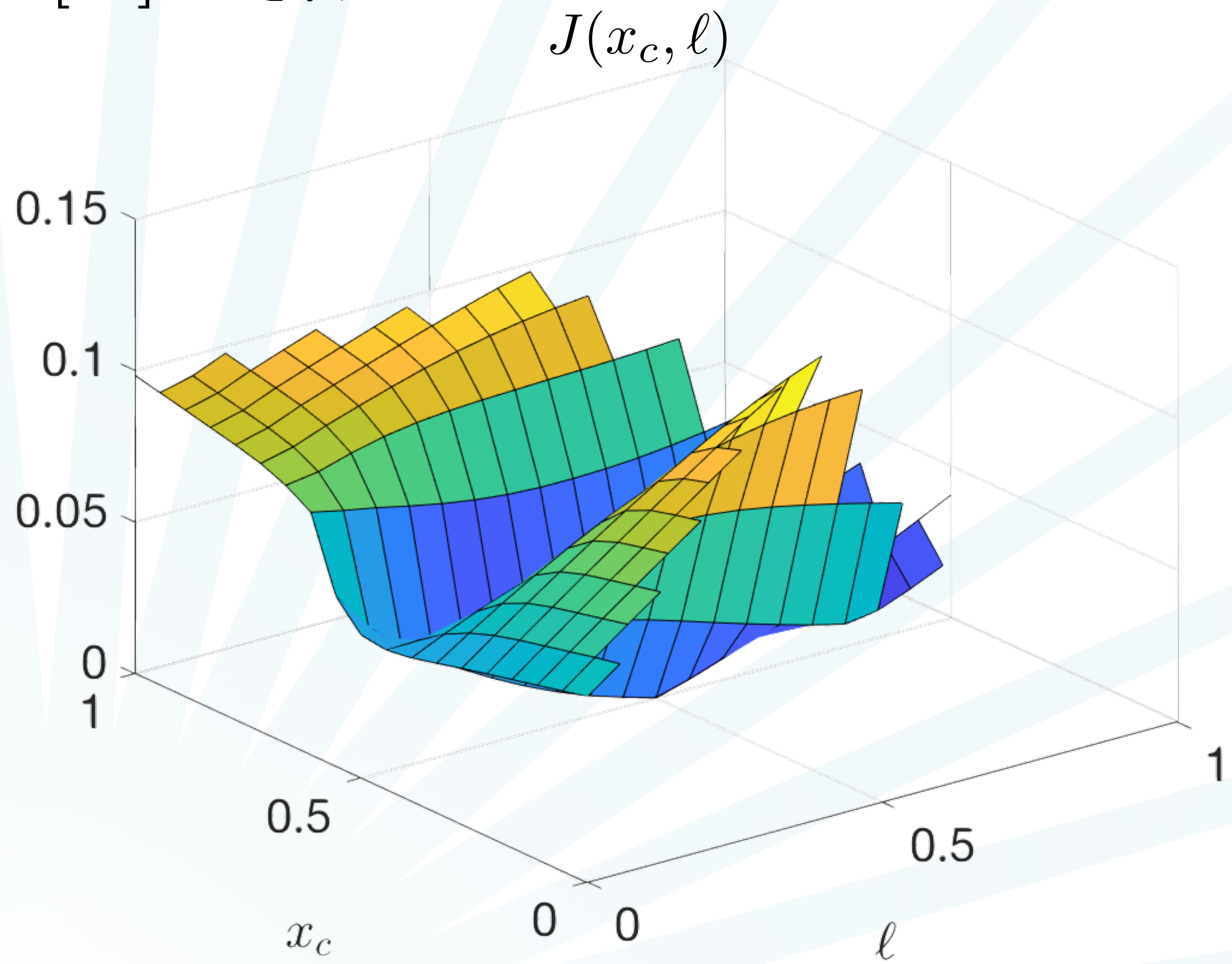
Boundary conditions:

- inlet: enthalpy H_L and total pressure $p_{tot,L}$
- outlet: pressure p_R

$$\begin{cases} H &= E + \frac{p}{\rho}, \\ p &= (\gamma - 1)\left(\rho E - \frac{1}{2}\rho u^2\right), \\ p_{tot} &= p + \frac{1}{2}\rho u^2. \end{cases}$$

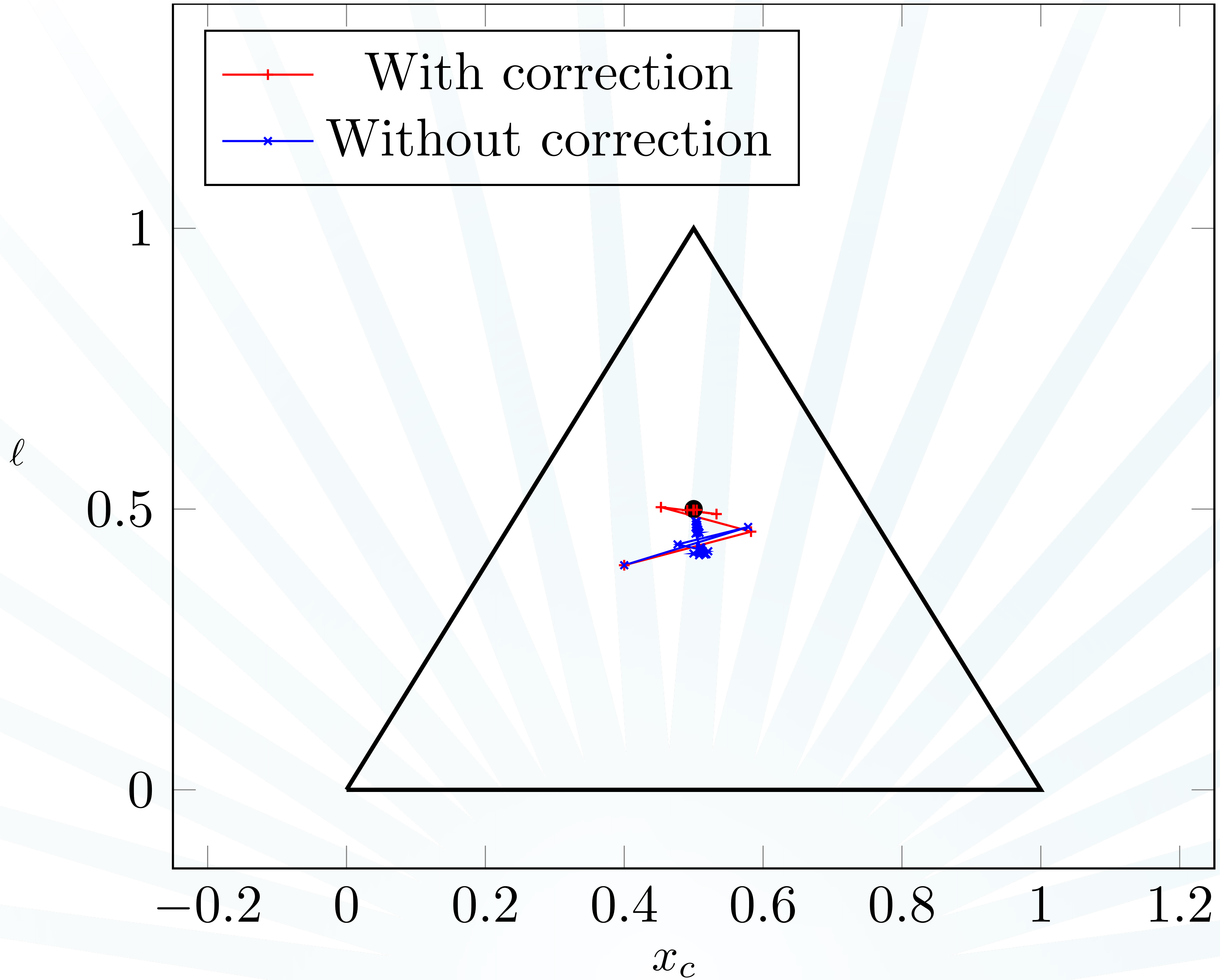
These b.c. provide a discontinuous solution [13] $\forall \mathbf{a} \in \mathcal{A}$.



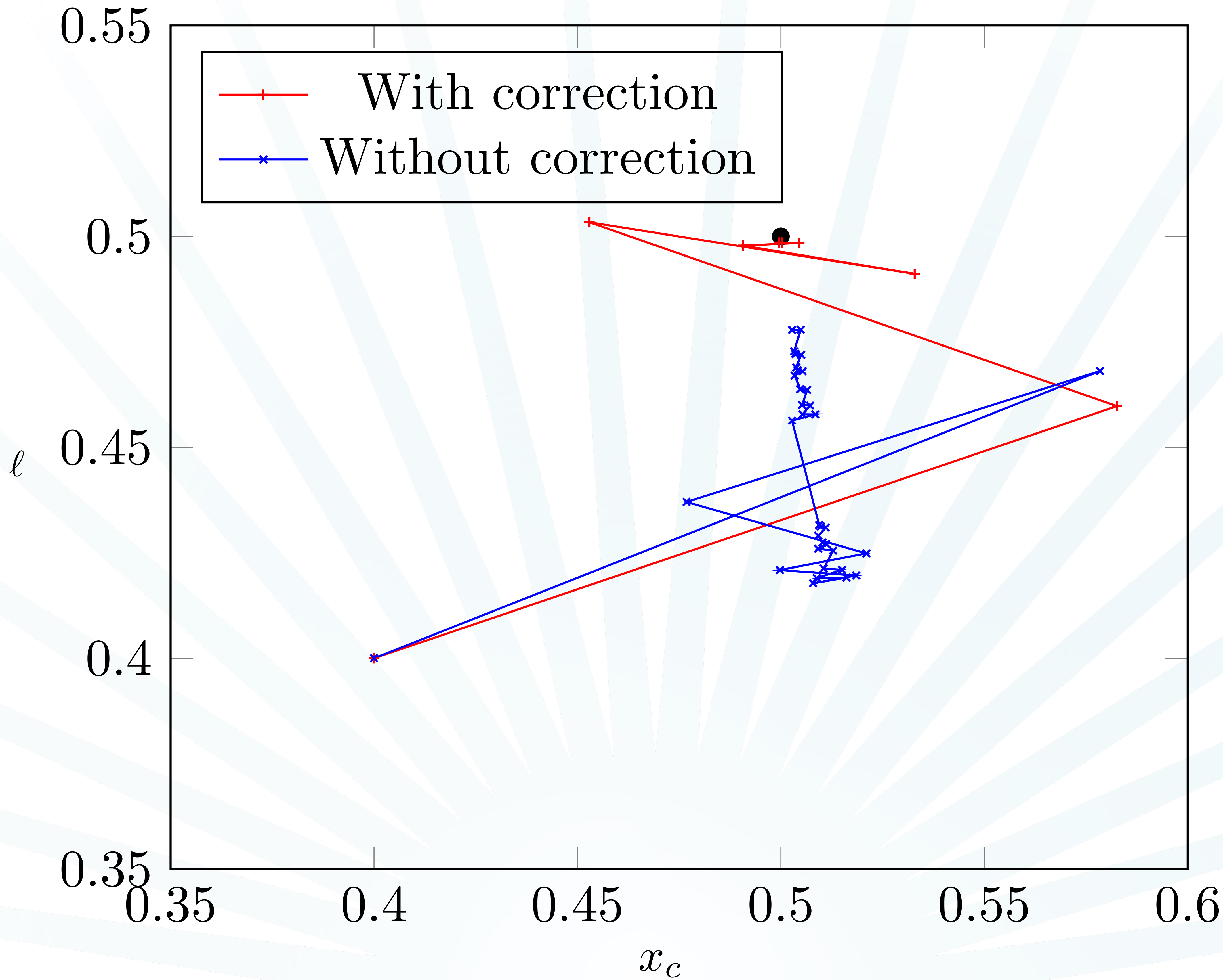


[13] Giles, M. B., & Pierce, N. A. (2001). Analytic adjoint solutions for the quasi-one-dimensional Euler equations. *Journal of Fluid Mechanics*, *426*, 327-345.

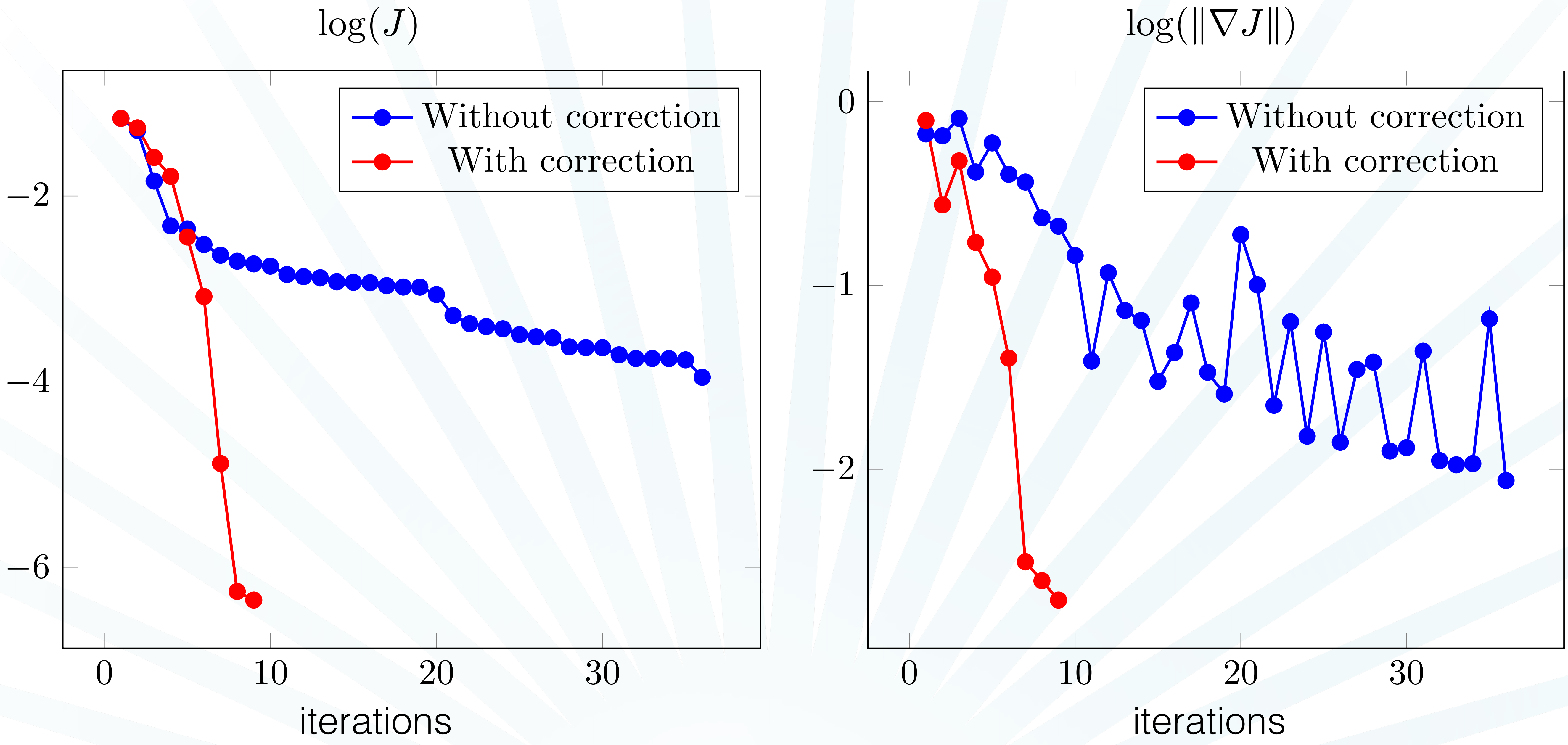
Optimization



Optimization



Optimization



Conclusion and perspectives

Conclusion:

- We defined a sensitivity system valid in case of discontinuous state
- The correction term is well defined
- The correction term is important in applications

Future development:

- Extension to non-classical shocks
- Effects of the numerical diffusion for the applications
- Extension to 2D

Thank you

for your attention!