Higher order schemes in time for the surface quasi-geostrophic system under location uncertainty

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SQG system under location uncertainty

LU framework: based on the following decomposition of the Lagrangian velocity in two components

$$\mathrm{d}\mathbf{X}_t = \mathbf{u}(\mathbf{X}_t, t)\mathrm{d}t + \sigma(\mathbf{X}_t, t)\mathrm{d}\mathbf{B}_t$$

one can compute the **stochastic transport operator**:

$$\mathbb{D}_t b := \mathrm{d}_t b + \mathbf{v}^* \cdot \nabla b \ \mathrm{d}t + \sigma \mathrm{d}\mathbf{B}_t \cdot \nabla b - \frac{1}{2}\nabla \cdot (a \nabla b)\mathrm{d}t,$$

where

$$\mathbf{v}^* = \mathbf{u} - \frac{1}{2}\nabla \cdot a - \sigma(\nabla \cdot \sigma)$$

Therefore, the surface quasi geostrophic system under location uncertainty is:

$$\begin{cases} \mathbb{D}_t b = 0, \\ b = N(-\Delta)^{1/2}\psi, \\ \mathbf{u} = \nabla^{\perp}\psi, \end{cases}$$

Towards the Milstein scheme

The main equation is:

$$b_t = b_{t_0} + \int_{t_0}^{t} \frac{1}{2}\nabla\cdot(a\nabla b) - \mathbf{v}^*\cdot\nabla b \,\mathrm{d}s - \int_{t_0}^{t} \nabla b\cdot\sigma\mathrm{d}\mathbf{B}_s,$$

$$\sigma\mathrm{d}\mathbf{B}_s \to \sum_m \varphi_m \mathrm{d}\beta_s^m,$$

We model the noise by decomposing it onto a basis using a POD approach

We then define the following functions:

$$f(b_t,t) = \frac{1}{2}\nabla\cdot(a\nabla b) - \mathbf{v}^*\cdot\nabla b \qquad g^m(b_t,t) = \nabla b\cdot\varphi_m$$

We can apply Itō formula for f and g^m, obtaining:

$$f(b_t,t) = f(b_{t_0},t_0) + \int_{t_0}^{t} \underbrace{\frac{\partial f}{\partial s}}_{=0}(b_s,s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial f}{\partial b}(b_s,s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 f}{\partial b^2}(b_s,s)\mathrm{d}\langle b,b\rangle_s$$

$$g^m(b_t,t) = g^m(b_{t_0},t_0) + \int_{t_0}^{t} \underbrace{\frac{\partial g^m}{\partial s}}_{=0}(b_s,s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial g^m}{\partial b}(b_s,s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 g^m}{\partial b^2}(b_s,s)\mathrm{d}\langle b,b\rangle_s$$

Milstein scheme

By replacing everything in the Itō formulas and then into the main equation, one finds:

$$b_t = b_{t_0} + f(b_{t_0})\Delta t - \sum_m g^m(b_{t_0})\Delta\beta^m + \int_{t_0}^{t}\int_{t_0}^{s}\sum_{m,k} g^m(g^k(b_\tau))\mathrm{d}\beta^k_\tau \mathrm{d}\beta^m_s \tag{1}$$

Euler-Maruyama

We define the following quantities:

$$G^{m,k} := g^m(g^k(b_{t_0})) \qquad I^{m,k} := \int_{t_0}^{t}\int_{t_0}^{s} \mathrm{d}\beta^k_\tau \mathrm{d}\beta^m_s$$

Then the double integral in (1) can be approximated with:

$$\sum_{m,k} G^{m,k} I^{m,k} = \sum_{m,k} G^{m,k}\frac{I^{m,k}+I^{k,m}}{2} + G^{m,k}\frac{I^{m,k}-I^{k,m}}{2}$$

where $\frac{I^{m,k}+I^{k,m}}{2}$: $= \Delta\beta^m\Delta\beta^k - \delta_{m,k}\Delta t$

Lévy area, which can be simulated → weak approximation / neglected

Remark: if G is symmetric (i.e. $G^{m,k} = G^{k,m}$), then the Lévy area is not necessary:

$$\sum_{m,k} G^{m,k} I^{m,k} = \frac{1}{2}\sum_{m,k} G^{m,k} I^{m,k} + G^{k,m} I^{k,m} = \sum_{m,k} G^{m,k}\frac{I^{m,k}+I^{k,m}}{2}$$

Multi-step scheme

The final aim being to use Milstein scheme in a multi-step Runge-Kutta type method, we started studying Runge-Kutta methods in the stochastic framework, starting with SSPRK3 [1] and Heun [2].

First, we rewrite the system in Stratonovich form:

$$\begin{cases} \mathrm{d}_t b = f_s(b,u) + g_s(b) \circ \mathrm{d}B_t \\ u = -\kappa \nabla^{\perp} \Delta^{-1/2} b =: \mathscr{H}(b) \end{cases}$$

SSPRK3 [1]

$$\begin{cases} b^{(1)} = b^n + f_s(b^n, u^n)\Delta t + g_s(b^n)\Delta B^n \\ u^{(1)} = \mathscr{H}(b^{(1)}) \\ b^{(2)} = \frac{3}{4} b^n + \frac{1}{4}\left(b^{(1)} + f_s(b^{(1)}, u^{(1)})\Delta t + g_s(b^{(1)})\Delta B^n\right) \\ u^{(2)} = \mathscr{H}(b^{(2)}) \\ b^{n+1} = \frac{1}{3} b^n + \frac{2}{3}\left(b^{(2)} + f_s(b^{(2)}, u^{(2)})\Delta t + g_s(b^{(2)})\Delta B^n\right) \end{cases}$$

Heun [2]

$$\begin{cases} b^{(1)} = b^n + f_s(b^n, u^n)\Delta t + g_s(b^n)\Delta B^n \\ u^{(1)} = \mathscr{H}(b^{(1)}) \\ b^{n+1} = \frac{1}{2} b^n + \frac{1}{2}\left(b^{(1)} + f_s(b^{(1)}, u^{(1)})\Delta t + g_s(b^{(1)})\Delta B^n\right) \end{cases}$$

[1] Numerically modeling stochastic Lie transport in fluid dynamics, Multiscale Modeling & Simulation 17.1 (2019): 192-232. C. Cotter, D. Crisan, D. Holm, W. Pan and I. Shevchenko.
[2] Modelling uncertainty using stochastic transport noise in a 2-layer quasi-geostrophic model. Foundations of Data Science, 2.2 (2020). C. Cotter, D. Crisan, D. Holm, W. Pan and I. Shevchenko.

Convergence

but we don't know the exact solution

$$\mathbb{E}\left[\|b(T,\mathbf{x}) - b_h(n\Delta t,\mathbf{x})\|\right] \leq C\Delta t^{\gamma}$$

We want to estimate the rate of convergence

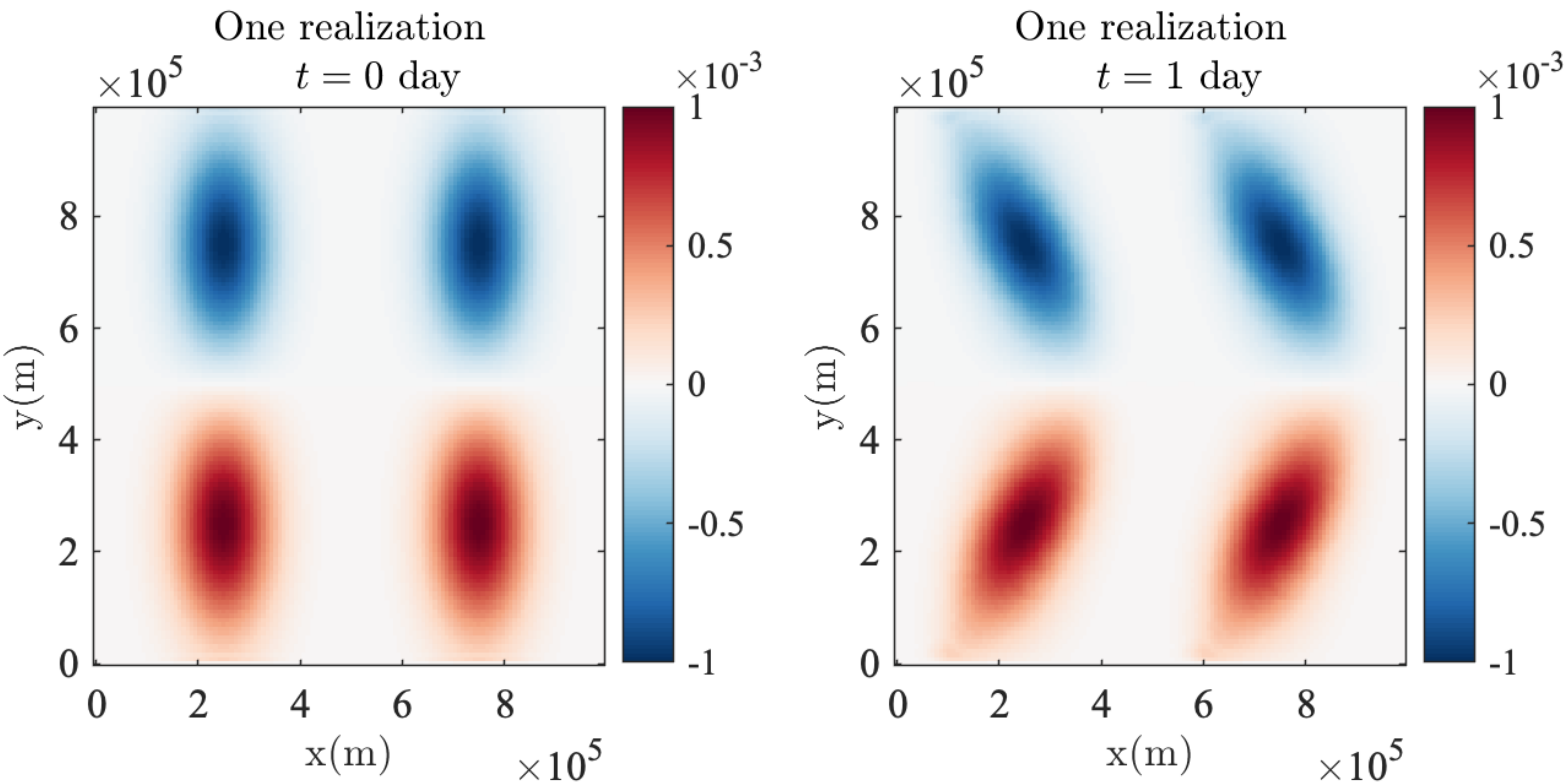
With three ("small enough") values of Δt one can provide the following estimate of γ

$$\gamma = \log_2\left(\frac{e_1}{e_2}\right) \quad e_1 := \mathbb{E}\left[\left|\tilde{S}(T,\Delta t) - \tilde{S}\left(T,\frac{\Delta t}{2}\right)\right|\right] \quad e_2 := \mathbb{E}\left[\left|\tilde{S}\left(T,\frac{\Delta t}{2}\right) - \tilde{S}\left(T,\frac{\Delta t}{4}\right)\right|\right]$$

- We used a fixed spatial mesh of 128x128
- We chose Δt = 60, 120, 240s
- 100 samples
- 1 day of simulation

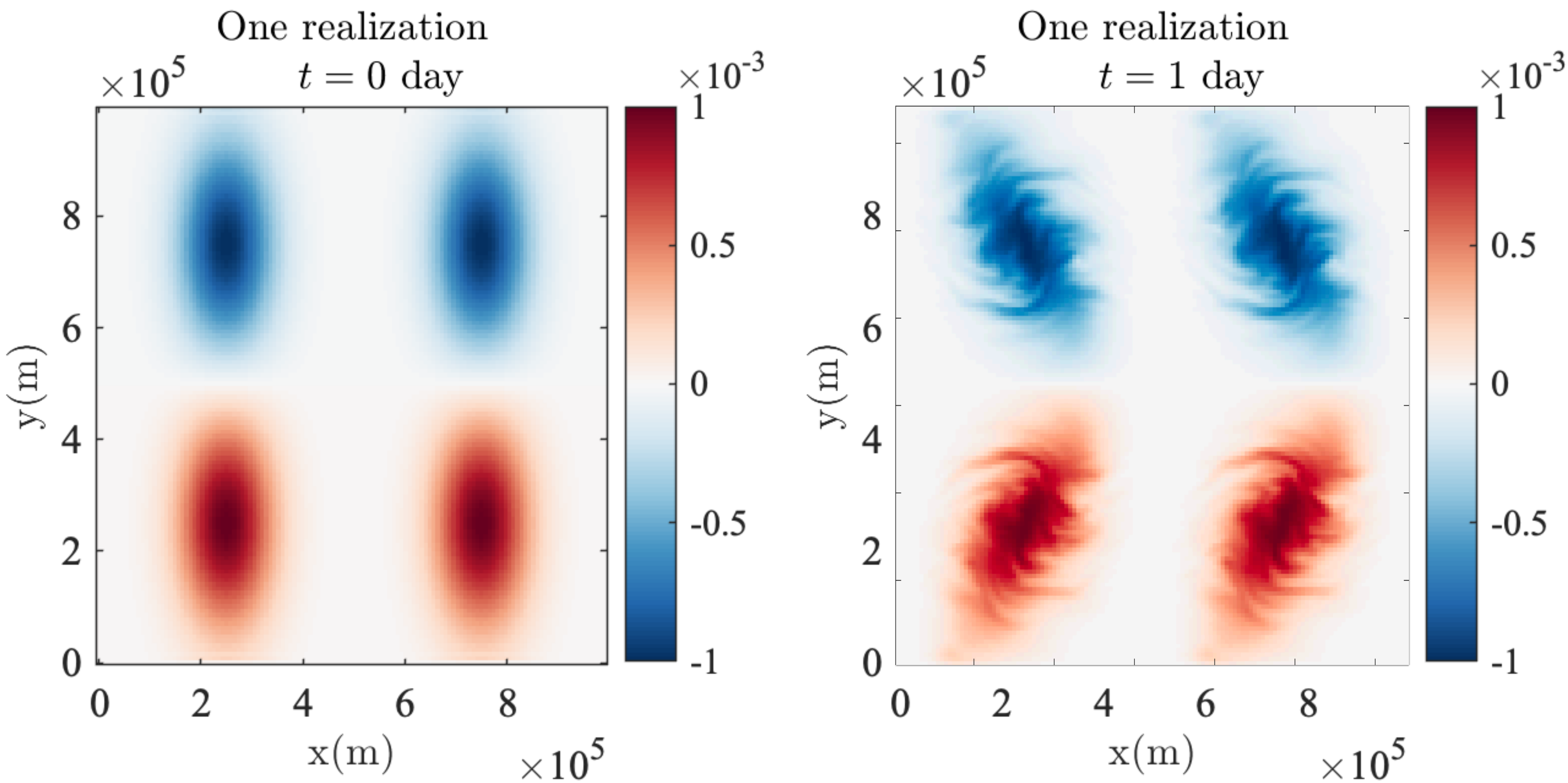
Numerical results - noise x1

Euler Maruyama



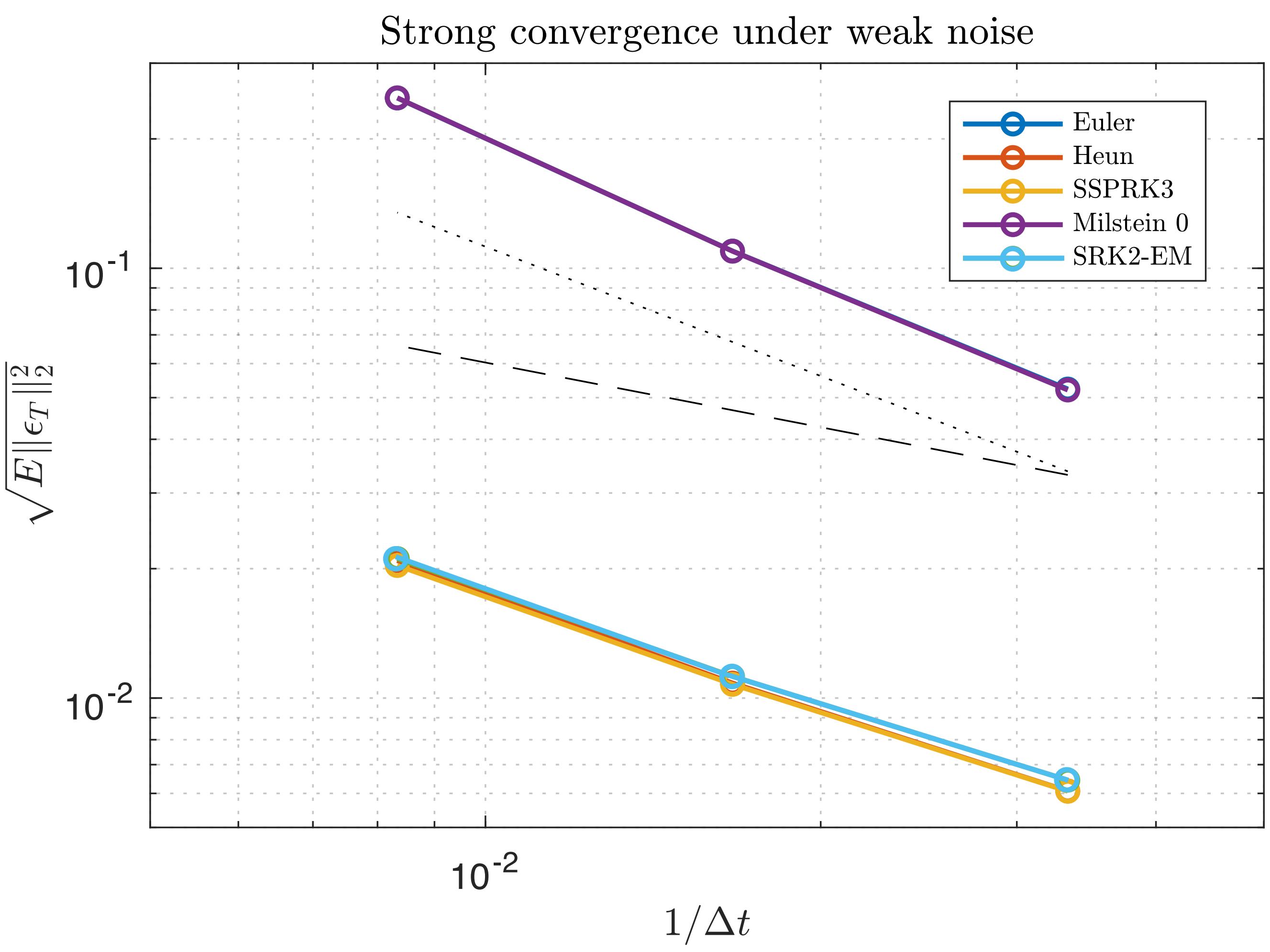
Numerical results - noise x10

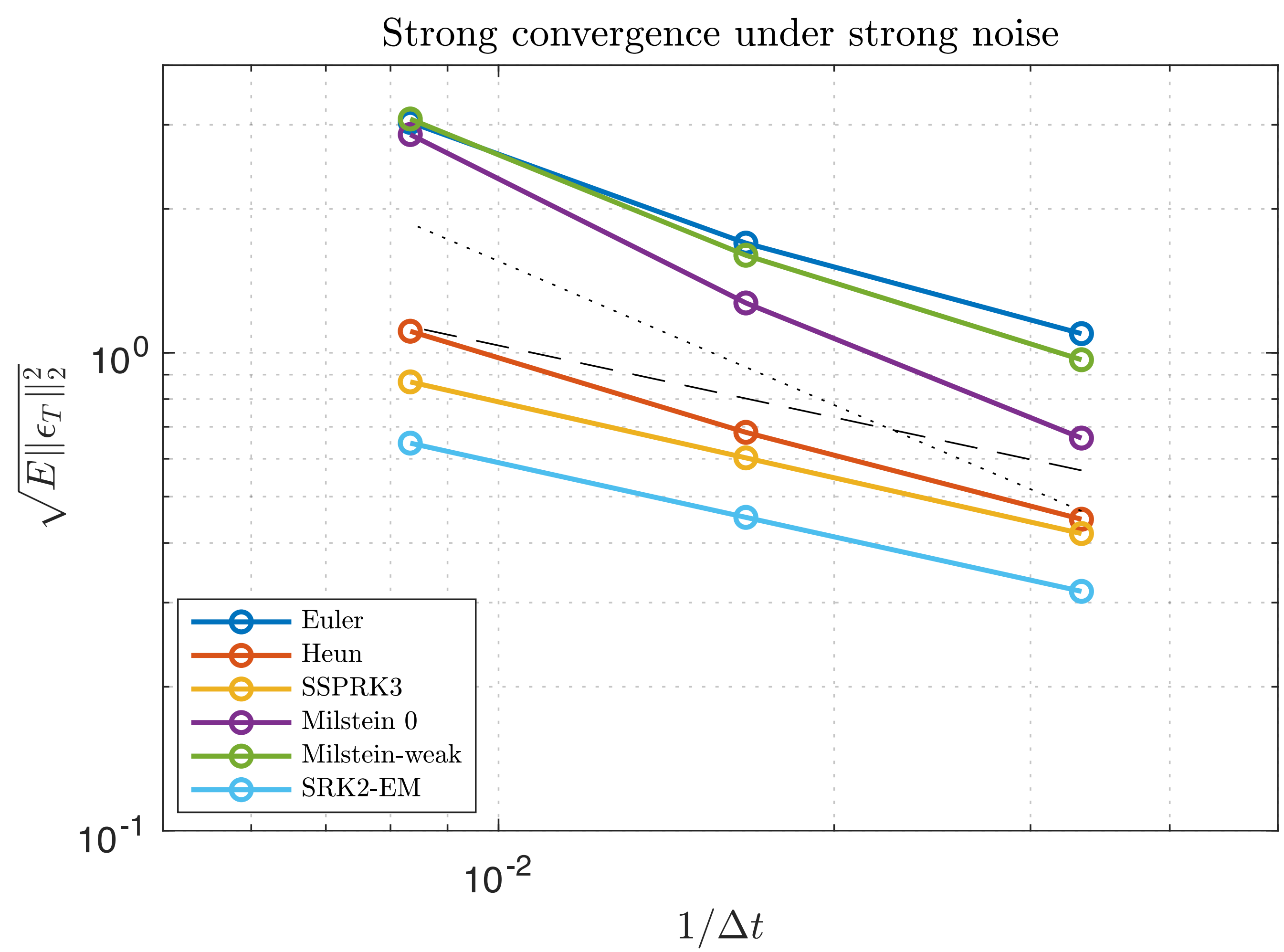
Euler Maruyama



Numerical results

$$\mathbb{E}\left[\|b(T,\mathbf{x}) - b_h(n\Delta t,\mathbf{x})\|\right] \leq C\Delta t^{\gamma}$$





Conclusion and perspectives

Conclusion

- Milstein schemes improve the numerical results, in particular when used in a multi-step framework;
- The Lévy area does not seem to play a key role in these test cases, which allows us to drastically reduce the computational costs;
- Under weak noise, all the schemes tested provide very similar results.

Perspectives

- Understand if the (non) importance of the Lévy area is related to the test case, the equations, or other factors;
- Apply this numerical scheme to other equations, starting with barotropic QG.